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Sampling and Interpolation of Periodic Signals*

Zdravko Nikolov, Ruska Iinbor, Chavdar Korsemov, Hristo Toshev

Institute of Information Technologies, 1113 Sofia
E-mails: znikolov@iit.bas.bg rdimitrova@iit.bas.bg korsemov@excite.com

Abstract: *The present paper discusses the sampling and restoration of periodic signals using systems of evenly shifted analyzing and synthesizing functions. The influence of these functions on the approximation accuracy as well as the requirements towards them concerning optimization of the error obtained, are shown. Some methods are considered for perfect interpolation of periodic frequency-limited functions. For such functions the interpolation with sincx functions, differing from the non-periodic ones, can be reduced to a finite number of summations. The methods for achieving computing stability are depicted.*

Keywords: *synthesizing and analyzing functions, bi-orthogonal functions, approximation accuracy*

1. Introduction

The interpolation and sampling are main procedures in signals digital processing. They are used not so intensely for the representation and restoration of continuous functions, as for their processing – alteration of the scale, deriving of desired parameters, realization of different geometric transformations such as translation, rotation, scale alteration, presenting with a different degree of resolution, etc.

The approximation accompanies every signal processing. Its participation is most sensitive in re-filtration, sampling and interpolation. The first two processes cause unrestorable alterations in the real continuous function. With a continuous function already presented in a discrete form, the interpolation may be executed with the help of different approaches depending on the purpose of processing. It however has its limited capacities with regard to the proximity to the initial continuous function. Moreover, it will be exposed that the requirements towards the functions for interpolation are stricter than those for re-filtration.

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The question for quantitative evaluation of the changes introduced in interpolation gains particular importance. It has been regarded rather in its qualitative than in the quantitative aspect until recently. In the last several years a great number of researchers have developed tools for defining these changes, most often considered as noise components. They play a significant role in the geometric transformations in computer tomography.

II. Presentation and approximation

The classical signal processing assumes the paradigm that the continuous signal can be presented in the form of discrete convolution of the type:

(1) $$\tilde{s}(t) = \sum_k a_k \varphi(t-k),$$

where a_k are measurers of the continuous function $s(t)$, and $\{\varphi_k = \varphi(t-k)\}$ is a set of translated functions at equidistant intervals. The wavelet theory shows that all the orthogonal scale functions [2] can be used as such interpolation functions.

In a stricter sense in order that $\{\varphi_k\}$ can form a linear subspace in Hilbert space H, i.e. $V_\varphi = \text{span}\{\varphi_k\} \in H$, it is necessary $\{\varphi_k\}$ to form Reisz basis, the set of linearly independent vectors $\{\varphi_k, k \in Z\}$, if the double inequality below given is satisfied:

$$C_1^2 \|a_i\| \le \left\| \sum_i a_i \varphi(t-i) \right\|^2 \le C_2^2 \|a_i\|^2 \quad \forall \{a_i, i \in Z\} \subset l_Z^2,$$

$$0 < C_1 \le C_2 < \infty.$$

At $C_1 = C_2$, φ_i form an orthonormal basis

(2) $$\langle \varphi_i, \varphi_k \rangle = \langle \varphi(t-i), \varphi(t-k) \rangle = \delta(i-k).$$

The upper constraint can be reduced to the condition:

(3) $$C_1^2 \le \sigma(\omega) \le C_2^2, \text{ where } \sigma(\omega) = \sum_i \left|\hat{\varphi}(\omega + 2\pi k)\right|^2,$$

$\hat{\varphi}(\omega) = \Im[\varphi(t)]$, $\Im$ is an operator of the Fourier transform.

This condition can be shown as follows:

Let $s(t) = \sum_k a_k \varphi(t-k)$ and $s(t) \subset L_R^2$,

$$\hat{S}(\omega) = \sum_k a_k e^{-jk\omega} \hat{\varphi}(\omega), \quad A(\omega) = \sum_k a_k e^{-jk\omega},$$

$$\|S(t)\|^2 = \|S(\omega)\|^2 = \int_{-\infty}^{+\infty} |A(\omega)|^2 |\hat{\varphi}(\omega)|^2 df,$$

$$A(\omega + 2\pi k) = \sum_k a_k e^{-j\omega} e^{-j2\pi k} = A(\omega),$$

$$\|S(\omega)\|^2 = \int_0^{2\pi} |A(\omega)|^2 |\varphi(\omega)|^2 df + \int_0^{2\pi} |A(\omega+2\pi)|^2 |\hat{\varphi}(\omega+2\pi)|^2 + \ldots,$$

$$\|S(\omega)\|^2 = \sum_k \int |A(\omega+2\pi k)|^2 |\hat{\varphi}(\omega+2\pi k|^2 df =$$

$$\int_0^{2\pi} |A(\omega)|^2 \sum |\hat{\varphi}(\omega+2\pi k)|^2 df = \int_0^{2\pi} |A(\omega)|^2 \sigma(\omega) df .$$

At $C_1^2 \le \sigma(\omega) \le C_2^2,\ \forall \omega \subset 0 \div 2\pi$

$$C_1^2 \|A(\omega)\|^2 \le \|\hat{S}(\omega)\|^2 \le C_2^2 \|A(\omega)\|^2 .$$

The "measures" $\{a_k\}$ of function $S(t)$ can be obtained either with the help of a convolution with another function for the moments of k, or projecting on the set of other analyzing functions, also forming a linear vector subspace in $H\{\theta(t-k)\}$

$$a_k = \langle s(t), \theta(t-k) \rangle .$$

Let P and R denote respectively the operations projection and interpolation (reconstruction).

It is obvious that P transforms the space L_R^2 into l_z^2, while the reconstruction inversely transforms l_z^2 into L_{R}^2. The signal reconstructed has the form:

$$\tilde{S}(t) = \mathrm{RP}\, S(t) = \sum_k \int s(\tau).\theta^*(\tau-k)\varphi(t-k)d\tau.$$

Setting $K(t,\tau) = \sum_k \theta^*(\tau-k)\varphi(t-k)$

(4) $$\tilde{s}(t) = \int \tilde{s}(\tau) K(t,\tau) d\tau .$$

The kernel $K(t,\ \tau)$, regarded as a discrete function with parameters t and τ represents a correlation function between the analyzing and the synthesizing function.

A direct approach to increase the validity of the transformation is to decrease the sampling step. This can be analytically expressed replacing the integer variable n by αn, $\alpha > 0$. Then the expressions become:

(5) $$\varphi_\alpha(t-n) = \frac{1}{\sqrt{\alpha}} \varphi\left(\frac{t}{\alpha} - n\right),\ \theta_\alpha(t-n) = \frac{1}{\sqrt{\alpha}} \theta\left(\frac{t}{\alpha} - n\right),$$

(6) $$T_\alpha [S(t)] = \mathrm{P}_\alpha.\mathrm{R}_\alpha [S(t)] = \frac{1}{\alpha} \int K\left(\frac{t}{\alpha}, \frac{\tau}{\alpha}\right) S(\tau) d\tau .$$

There exist large classes of functions $\varphi(t)$ and $\theta(t)$, for which

$$\text{at } \alpha \to 0, \quad \|P_\alpha R_\alpha S(t) - S(t)\| \to 0 .$$

For the transformation kernel $K(t,\ \tau)$ I v e s M e y e r [1] has shown that if it is periodical with a period 1, $K(t,\tau) = K(t+1,\tau+1)$, the following conditions are satisfied:

$$\left|K(t,\tau) \le \frac{C_0}{1+(t-\tau)^2}\right|, \int K(t,\tau)d\tau = 1 \ \ \forall t \in R.$$

Then

$$\forall s(t) \in L_R^2, \ \left\|T_\alpha[s(t)] - s(t)\right\|^2 \to 0 \text{ at } \alpha \to 0.$$

The condition $\int K(t,\tau)d\tau = 1$ implies the constraint $\int \theta(t)dt \ne 0$. Indeed,

$$\int K(t,\tau)d\tau = \int \theta^*(\tau - k)d\tau \sum \varphi(t-k)d\tau.$$

If assumed that $\int \theta(t)dt = 1$, which is easily fulfilled, the constraint $\sum \varphi(t-k) = 1$ is obtained, after introducing an appropriate multiplier, which in the frequency domain will lead to:

(7) $$\hat{\varphi}(0) = 1, \varphi(2\pi k) = 0, \forall k \in Z, k \ne 0.$$

This important condition is shown by [2]. The placing of additional constraints on $\varphi(t)$ and $\theta(j)$ is possible. One significant constraint is the condition for biorthogonality – to belong to one and the same vector space and to be mutually orthogonal, according to the relation:

$$\langle \varphi(t-k), \theta(t-j) \rangle = \delta(k-j).$$

In this case the re-filtration is reduced to orthogonal projection and the error obtained at twofold transform is minimal in the mean square sense.

The operator projection has an important property $\mathrm{P}^n = \mathrm{P} \quad \forall n \in Z$. When selecting a synthesizing function $\varphi(t)$, the requirement for bi-orthogonality imposes the unambiguous determination of the analyzing $\theta(t)$:

$$\theta(t) = \sum_i a_i \varphi(t-i), \ \hat{\theta}(\omega) = \hat{\varphi}(\omega) \sum_i a_i e^{-j\omega i} = \hat{\varphi}(\omega) \hat{m}(\omega)$$

$$\langle \theta(t), \varphi(t-l) \rangle = \int_{-\infty}^{\infty} \theta(t)\varphi^*(t-l)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{\varphi}(\omega)\right|^2 e^{j\omega l} \sum_k a_k e^{-j\omega k} d\omega = \delta(l).$$

$$m(\omega) = \sum_k a_k e^{-j\omega k}$$

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{\varphi}(\omega)\right|^2 m(\omega) e^{j\omega l} d\omega = \frac{1}{2\pi}\int_0^{2\pi} \sum \left|\hat{\varphi}(\omega + 2\pi n)\right|^2 m(\omega) e^{j\omega l} d\omega, .$$

$m(\omega) = m(\omega + 2\pi i)$ is a periodic function with a period 2π.

$$\sigma(\omega) = \sum_n \left|\hat{\varphi}(\omega + 2\pi n)\right|^2,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sigma(\omega).m(\omega).e^{j\omega l} d\omega = \delta(l).$$

The orthogonality condition is satisfied at $\sigma(\omega)\hat{m}(\omega)=1$,

$$\sigma(\omega)\frac{\hat{\theta}(\omega)}{\hat{\varphi}(\omega)}=1,$$

(8)
$$\hat{\theta}(\omega)=\frac{\varphi(\omega)}{\sigma(\omega)}.$$

The pair of bi-ortogonal functions $\varphi(t)$ and $\theta(t)$ thus obtained, connected with relation (8) ensures the best restoration in the mean-square meaning.

Of course, this condition for orthogonality is not obligatory. It is possible to choose the analyzing function with respect to other requirements, for example – simplicity, computing efficiency, causality, preset components, etc.

Let us consider the classical synthesizing function

$$\varphi(x)=\operatorname{sinc} x=\frac{\sin\pi x}{\pi x}, \qquad \hat{\varphi}(\omega)=\begin{cases}1, \text{ if } 0\le|\omega|\le\pi\\ 0, \text{ if } |\omega|>\pi\end{cases}.$$

The set $\left\{\frac{\sin(\pi x-k)}{\pi x-k}, k\in Z\right\}$ forms a complete orthogonal basis in Hilbert space X. The vector subspace V_{sinc} = span sinc$x$ contains all frequency limited functions. The spectrum $\hat{S}(\omega)=0$, for $\forall\omega>\pi$.

$\sigma(\omega)=\sum_k|\varphi(\omega+2\pi k)|^2=1$, since $\hat{\varphi}(\omega)=1\ \forall|\omega|<\pi$, and $\hat{\theta}(\omega)=\hat{\varphi}(\omega)$ – the synthesizing and analyzing fucntions coincide.

The “measures” s_k for the signal $s(x)$ are defined by the projecting:

$$s_k=\left\langle s(x),\frac{\sin(\pi x-k)}{\pi x-k}\right\rangle.$$

Since $\varphi(x)$ is an even function, the projection and convolution will give one and the same result:

$$s_k=\left\langle s(x,\varphi_k(x)\right\rangle=s(x)\varphi(x)\Big|_{x=\frac{k}{\pi}}.$$

In this way the coefficients s_k can be obtained as accounts at the output of a filter with a pulse function $\varphi(t)$ through intervals Δ= 1/π, as shown in Fig. 1.

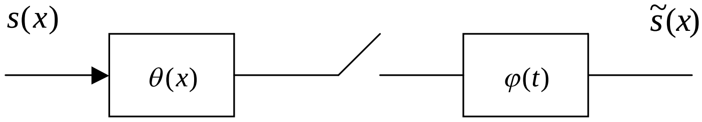

Fig. 1

In case that $s \in V_{\mathrm{sinc}}$, i.e. $s(x) = \sum a_i \sin c\,(x - i)$, and since sincπ = 0 $\forall n \in Z, n \neq 0$, $a_i = s(i)$, and the analyzing function $\theta(x) = \delta(n)$.

By the way, $\theta(x) = \delta(n)$ for each cardinal synthesizing function $\varphi(x)$, if the approximating function belongs to the vector subspace generated by it.

III. Approximation error

The discretization has the purpose to present the continuous function $s(x)$ by a row of data without loss of the information contained in it. This means that the restoration of the initial function with a preset accuracy will be possible. We shall consider the presentation of $s(x) \in L^2_{\mathrm{R}}$ from a point in the vector space $V_\varphi = \mathrm{span}\,(\varphi(x - i), i \in Z)$, formed by evenly translated functions $\varphi(x - i)$. These functions are linearly independent, forming Reisz' basis in Hilbert space H. In this way the function $s(x)$ is represented by:

$$s_1 = \sum_i a_i \varphi(x - i) = A^{\mathrm{T}} \phi(x),$$

$$A = \left| a_0 a_1 \ldots a_N \right|^{\mathrm{T}} \quad \phi(x) = \left| \varphi(x)\varphi(x-1)\ldots\varphi(x-N) \right|^{\mathrm{T}}. \tag{9}$$

The vector A represents the coordinates of the signal approximated in V_φ. As evident, (9) can be regarded as discrete convolution. Each coordinate a_i is the "measure", the "content" of $s(x)$ in $\varphi(x - i)$. One way to define A is projecting $s(x)$ on V_φ, thus ensuring the smallest difference between s and s_1 in the mean-square meaning. The projection is reduced to the determination of scalar products $a_i = \left\langle s(x), \theta_p(x - i) \right\rangle$, where $\theta_p(x)$ is an analyzing function, the translated versions of which form also a vector space, belonging to V_φ. The condition for projection defines unambiguously $\theta(x)$ at selected basis $\{\varphi_i\}$. In the frequency domain it gets the form:

$$\hat{\theta}(\omega) = \frac{\hat{\varphi}(\omega)}{\sigma(\omega)}, \quad \sigma(\omega) = \sum_n \left| \varphi(\omega + 2\pi n) \right|^2 .$$

In these correlations $\varphi(x)$ and $\theta_p(x)$ are mutually orthogonal and form a bi-orthogonal pair for analysis and synthesis. The operation scalar product is often replaced by convolution and hence it is presented as filtration, usually defined by re-filtration. The operation with an analyzing function usually transforms the real space of $s(x)$ into V_φ.

In most of the cases the determination of A by the bi-orthogonal function $\theta p(x)$ is too resource consuming and hence operation with a simplified analyzing function is preferred. Still more, there exist many functions giving presentations close to the optimal ones.

One evident approach to improve the approximation is at selected f to decrease the sampling step along αz, $0 < \alpha < 1$, instead of along z. To compare the approximation quality for different $\varphi(x)$, the speed of error decrease with the reduction of step α can be used.

Let η_p denotes the error at projection with the minimal in the mean-square sense $\eta_p = s(x) - \mathrm{T}s(x)$.

Here T = RP is the projection, following the synthesizing operator (P) one (R).

A fundamental result for the approximation error, obtained by [2] shows that $\|\eta_{p\alpha}\|_{L_2} \sim \alpha^L$, if $\hat{\varphi}(0) \neq 0$ and $\varphi^{(k)}(2\pi n) = 0, \forall n \neq 0$ and k = 0, 1, ..., L–1. Here $\varphi^{(k)}(\omega)$ is k-th derivative.

This relation between the convergence speed of $\|\eta_\alpha\|$ and the properties of $\varphi(x)$ can be represented in other, equivalent ways.

The speed is kept L, if $\varphi(x)$ has L null moments. The fundamental requirement for a convergence speed L is that the space of reconstruction contains polynomials of $L - 1$ degree.

At orthogonal projection, this degree is determined by $\varphi(x)$ and the error is minimal in the mean-square meaning. At non-orthogonal projection however, the error is not only non-minimal, but the convergence degree can be lower. Under certain constraints on the analyzing function $\theta(x)$, the convergence degree can be preserved as defined by $\varphi(x)$. In [2] the pairs of functions $\varphi(x)$ and $\theta(x)$ are defined as quasi-biorthogonal, if the convergence degree of $\|\eta_\alpha\|$ remains L, as determined for an optimal analyzing function.

The condition below given is accepted as a common requirement for quasi-biorthogonality:

$$\int_{-\infty}^{\infty} x^k \theta(x) dx = \int_{-\infty}^{\infty} x^k \theta_p(x) dx < \infty \quad \forall k = 0, 1, ..., L-1 \cdot$$

Here $\theta_p(x)$ forms a biorthogonal pair with $\varphi(x)$.

The requirements towards $\theta(x)$ for quasi-biorthogonality can be determined in a different way, but as a rule they give a comparatively wide freedom of choice. This enables the finding of efficient algorithms of approximation.

In [2, 3, 4] the results from some serious investigations on the errors occurring in approximation are exposed.

A significant theorem is proved in [2], that the error from interpolation of the continuous function $s(x) \in W_2^r$ with r>1/2 is determined by the expression:

(10) $$\|\eta_\alpha(s)\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{S}(\omega)|^2 E(\alpha\,\omega) d\omega + (\varepsilon(s,\alpha))^2,$$

where $\eta_\alpha(s) = s - \mathrm{T}_\alpha s$,

(11) $$E(\alpha\,\omega) = 1 - \frac{|\hat{\varphi}(\omega)|^2}{\sigma_\varphi(\omega)} + \sigma_\varphi(\omega) \left|\hat{\theta}(\omega) - \hat{\theta}_p(\omega)\right|^2,$$

$\hat{\theta}_p(\omega)$ is the biorthogonal pair of $\hat{\varphi}(\omega)$.

$\hat{\theta}(\omega)$ is the actually used analyzing function.

$$(12) \qquad \left|\varepsilon(s,\alpha)\right| \le K\alpha\left\|s^{(r)}\right\|_{L^2}, \; K = \frac{1}{\pi^{r}}\sqrt{\zeta\,(2r)\left\|E(\alpha\,\omega)\right\|_{\infty}}\,,$$

$\zeta\,(2r) = \sum_{n=1}^{\infty}\frac{1}{n^{2r}}$ is Riman ζ-function,

$$s(x) \in W_2^r \text{ if } \int (1+\omega^2)^r \left|\tilde{S}(\omega)\right|^2 d\omega < \infty\,.$$

IV. Interpolation of periodic signals

The interpolation of sampled periodic signals is interesting for different areas of signal processing – functions reconstruction, alteration of the sampling frequency, delays different than the sampling step, determining of extremums with high resolution.

The use of the periodicity, which converts the linear convolution to a circular one enables the development of specialized high efficient methods of interpolation. On its basis some methods are suggested in [5, 6, 7], equivalent to interpolation with sincx functions, but with a finite discrete convolution. In [8] a method is presented, equivalent to the perfect frequency-bounded interpolation, but with time translation in the frequency domain and an inverse Fourier transform.

The present paper discusses the main principles for perfect restoration of sampled periodic signals, their possibilities and various problems of the software realization. Some questions connected with evaluation of the accuracy achieved, the character of the occurring errors, are also considered.

Interpolation with sinc*x*

Let $s(t)$ be a periodic signal, presented by its sampled values $s_n = s(n\tau)$, measured at equal intervals τ, with a period $T_0 = N\tau$, divisible to the step τ, at that $N \in Z$ can be an even or non-even number. It is accepted that $s(t)$ is frequency limited and that τ, N respectively are selected in such a way, that they completely represent the signal.

For such a signal, the complete restoration is obtained by the expression:

$$(13) \qquad s(t) = \sum_{n=-\infty}^{+\infty} s_n \operatorname{sinc}\left(\frac{t}{\tau} - n\right) = \sum_{n=-\infty}^{+\infty} \frac{\sin\left(\pi\frac{t}{\tau} - \pi n\right)}{\pi\left(\frac{t}{\tau} - n\right)}\,.$$

The function cardinal sinus has the form: $\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$.

In order to simplify the writing down of the relations, we set $x = t/\tau$ and the expression for complete restoration becomes:

$$(14) \qquad s(t) = s(x\tau) = \frac{\sin \pi x}{\pi} \sum_{n=-\infty}^{+\infty} S_n \frac{(-1)^n}{x-n}\,.$$

If the axis t is divided into intervals with length $N\tau$, the expression $s(t)$ can be represented as:

$$s(t)=\frac{\sin\pi x}{\pi}\sum_{j=-\infty}^{+\infty}\sum_{n=L+jN}^{L+(j+1)N-1}S_n\frac{(-1)^n}{x-n},$$

$$s(t)=\frac{\sin\pi x}{\pi}\sum_{j=-\infty}^{+\infty}\sum_{n=0}^{N-1}S_{jN+n}\frac{(-1)^{jN+n}}{x-jN-n}.$$

Here L is conditionally accepted as the measuring start. For a periodic signal with a period $T_0 = N\tau$, $x_n = x_{n+jN}$, $j, n, N \in Z$.

It is evident from the last expression that the participation of $(-1)^{jN+n}$ makes the expressions for $x(t)$ dependent on the evenness or non-evenness of N. For even N:

$$s(t)=\frac{\sin\pi x}{N}\sum_{n=0}^{N-1}x_n(-1)^n\sum_{j=-\infty}^{+\infty}\frac{1}{\pi}\frac{1}{\frac{x-n}{N}-j}.$$

For non-even N:

$$s(t)=\frac{\sin\pi x}{N}\sum_{n=0}^{N-1}x_n(-1)^n\sum_{j=-\infty}^{+\infty}\frac{1}{\pi}\frac{(-1)^j}{\frac{x-n}{N}-j}.$$

The second multipliers in the last two expressions present the trigonometric functions respectively:

$$\operatorname{cotg}\pi\frac{x-n}{N}=\frac{\cos\pi\frac{x-n}{N}}{\sin\pi\frac{x-n}{N}}\quad\text{and}\quad\operatorname{cosec}\pi\frac{x-n}{N}=\frac{1}{\sin\pi\frac{x-n}{N}}$$

and the interpolation expressions take the form:

For even N –

(15) $$s(t)=\sum_{n=0}^{N-1}x_n\frac{\sin\pi(x-n)}{N\sin\pi\frac{x-n}{N}}.\cos\frac{x-k}{N};$$

and for non-even N –

(16) $$s(t)=\sum_{n=0}^{N-1}x_n.\frac{\sin\pi(x-n)}{N\sin\pi\frac{x-n}{N}}$$

The expressions (15) and (16) enable the complete restoration, a direct consequence of the interpolation with sinc x function, but with the significant advantage that it is completed in a finite number of operations.

In spite of the evident advantages, expressions (15) and (16) are unstable in a computing aspect due to the presence of trigonometric functions in the denominator. At each null, or values close to null, indeterminacies appear. It is recommendable, and sometimes possible to avoid the indeterminacies by some trigonometric transformations.

The trigonometric multiplier in (16) represents the famous Dirichlet kernel:

(17) $$D(x) = \frac{\sin \pi x}{N \sin \frac{\pi x}{N}}.$$

At non-even N, the interpolation for $s(t)$ is obtained as discrete convolution of x_n with $D(x)$:

(18) $$x(t) = \sum_{n=0}^{N-1} x_n D(x-n) = x_k D(x)\Big|_{x=\frac{t}{\tau}}$$

When introducing an interpolation kernel:

(19) $$D_1(x) = D(x).\cos\frac{\pi x}{N} = \frac{\sin \pi x}{N \sin \frac{\pi x}{N}}.\cos\frac{\pi x}{N};$$

the interpolation at even N obtains the simple form:

(20) $$x(t) = \sum_{n=0}^{N-1} x_k D_1(x-n) = x_k * D_1(x)\Big|_{x=\frac{t}{\tau}}.$$

The kernel $D_1(x)$ is in its essence the weighted by a weighting function $h(x) = \cos\frac{\pi x}{N}$ Dirichlet kernel (17). The weighting function is a positive semi-wave of a cosinus function with a period twice bigger than the period T of $x(t)$.

In spite of the simple expression, the problem of indeterminacy remains.

If the relation

(21) $$\sum_{k=0}^{k} \cos k2L = \frac{\sin(2k+1)L}{2\sin L} + \frac{1}{2}$$

be used, Dirichlet kernel gets the form:

(22) $$D(x) = \frac{1}{N}\left(2\sum_{k=0}^{N-1/2} \cos\frac{2\pi x}{N}k - 1\right).$$

If (21) is used again, the kernel $D_1(x)$ becomes:

(23) $$D_1(x) = \frac{1}{N}\left(\cos \pi x - 1 + 2\sum_{k=0}^{N/2} \cos\frac{2\pi x}{N}k\right).$$

When using $D(x)$ and $D_1(x)$ in the form (22) and (23), the interpolation is stable in a computing aspect.

V. Conclusion

The periodic signals are a specific class of signals. Their presentation with periodic basis functions upon an unlimited carrier, such as the complex harmonic functions, is considered appropriate. The restoration here exposed with a sin*cx* function is in fact transformed as restoration with harmonic functions.

It is interesting to note that the sampling itself can be directly adapted to such kind of representation, not to the traditional one with shifting of the basis function.

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Дискретизация и интерполация на периодични сигнали

Здравко Николов, Руска Ийнбор, Чавдар Корсемов, Христо Тошев

Институт по информационни технологии, 1113 София
E-mails: znikolov@iit.bas.bg rdimitrova@iit.bas.bg korsemov@excite.com

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В настоящата работа се разглежда дискретизирането и възстановяването на периодични сигнали с използването на системи от равномерно измествани анализиращи и синтезиращи функции. Показано е влиянието на тези функции върху точността на апроксимацията и изискванията към тях за оптимизиране на внасяната грешка. Показани са методи за съвършена интерполация на периодични честотно ограничени функции. За такива функции интерполацията с функции sinc*x*, за разлика от непериодичните, може да се сведе до краен брой сумирания. Показани са методите за постигане на изчислителна устойчивост.