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On the Multicriteria Integer Network Flow Problem

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Abstract*: The Pareto Optimal set for the multicriteria network flow problem is described by the theorem for ParetoOptimal flow augmenting paths. An approach for determining of the unsupported integer PO solutions of the investigated problem is proposed.*

Keywords: *multicriteria network flow problem, Pareto optimal solutions, multicriteria shortest path.*

1. Introduction

A flow in a network is defined as the function: X, $X \in X(G,v,c)$, where $X(G,v,c)$ is defined by the linear constraints (1)-(2) on the graph $G = \{N, U\}$; $N = \{1, 2, \dots, n\}$ is a set of nodes and $U = \{(i, j): i, j \in N\}, |U| = m$ – a set of arcs:

(1)
$$X(G,v,c): \quad \sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} v & \text{if } i = s, \\ 0 & \text{if } i \neq s,t, \\ -v & \text{if } i = t; \end{cases}$$

(2)
$$0 \le x_{ij} \le c_{ij,} \ (i, j) \in U.$$

The nodes $s = i_1 \in N$ and $t = j_1 \in N$, are called a source and a sink resp., c_{ij} denotes the capacities of the arcs $(i, j) \in U$ and v – the value of the feasible network flow. We denote $X = \{x_{ij}: (i, j) \in U\}$. If $X \in X(G, v, c)$, we also denote $X = X_v$.

The single-criterion problems for an optimal flow in a network are solved with the help of efficient polynomial algorithms that find an integer solution at integer arcs capacities. This property determines the separation of the flow problems in an independent class of linear programming problems.

Both the theoretical investigations and the practical applications require generalization of the problem considered.

As such generalizations may be considered the problem for an optimal flow with surrogate constraints (FSGP), when the set X is reduced by setting additional linear constraints on the values of the flow along the problem arcs, and also the multicriteria flow problem (MCF), in which improvement of the values of more than one linear criterion is required. Each Pareto optimal (PO) solution of MCF is a solution of the ε-constraint problem, which is a problem for minimizing only one of the criteria put on the set, which consists of the flow constraints and constraints on the values of the remaining criteria, i. e. FSGP. It has been proved for FSGP that it is equivalent to the general problem of linear programming. The integer properties are lost in it and also the possibility to solve it by flow methods. Some adaptations of the simplex methods are designed, which use the embedded flow structure in the constraints.

The essence of MCF problem is in the search for such a flow, usually with a fixed or maximal value, the criteria values of which satisfy the DM. If at a given stage of its solution it turns out that some criteria should not be still deteriorated, then the determination of a flow on the network with a value lower than the desired one, could become necessary.

MCF problem is a problem of linear multicriteria programming. The methods solving linear multicriteria problems can be applied to it and appropriately adapted with respect to the flow structure of the constraints. In this case MCF will be solved as a linear problem, the set of the basic PO integer solutions and its linear combinations will be defined, but this set will not comprise all integer solutions of MCF. The problem which determines all integer solutions of the multicriteria flow problem will be denoted as MCIF. Thorough analysis and review of the methods solving MCF and MCIF problems is proposed in [1].

The present paper examines the structure of the set of PO solutions of MCIF and a method for determinig integer PO solutions of the bicriteria flow problem is proposed.

2. Conditions for Pareto optimality of the multicriteria flow

Let us define the multicriteria problem MCF for a flow with a value v on graph G:

$$\text{MCF:}\quad \min{}^* \{ g_i (X), i \in I_k\},$$

$$\text{s. t.}$$

$$X \in X(G, v, c), \text{ where}$$

$$g_i(X) = \sum_{(i,j)\in U} a_{ij} x_{ij},$$

Lemma 1. Let p be a PO path in graph G. Let X_p be a feasible flow along the path p with a value of 1. Then X_p is a PO flow in G.

P r o o f:

Assuming the opposite, a contradiction is obtained with the Pareto optimality along the path p.

Theorem 1. Every integer PO solution of MCF with a value of the flow v may be presented as a sum of a PO solution of MCF with a value of the flow v –1 and a flow with a value 1 along a PO augmenting the flow path and vice versa.

P r o o f:

1. Let X_v be a PO solution of the problem MCF. Then there exist numbers $\lambda_i e \geq 0$, $\sum_{i \in I_k} \lambda_i = 1$, such that X_v is an optimal solution of the problem

$$\text{MF}(X):\ \min F(X) = \sum_{i \in I_k} \lambda_i\, g_i\,(X),$$
$$\text{s.t.}$$
$$X \in X(G, v, c).$$

The problem MF(X) is a single-criterion minimum flow problem. If we find the flow X_v by a minimal path method, then $X_v = X_{v-1} + X_p$.

Here $X_{v-1} = \{x_{ij}^{\,v-1}: (i, j) \in U\}$ is an optimal solution of the following problem:

$$\text{MF}(X):\ \min F(X) = \sum_{i \in I_k} \lambda_i\, g_i\,(X),$$
$$\text{s.t.}$$
$$X \in X(G, v-1, c),$$

i.e. it is a PO solution with a value of the flow v–1.

Hereby X_p is an optimal solution of the problem MF(X)

$$\text{MF}(X):\ \min F(X) = \sum_{i \in I_k} \lambda_i\, g_i\,(X),$$
$$\text{s.t.}$$
$$X(G_I(X_{v-1}), 1, 1):\quad \sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} 1, & \text{if } i = s, \\ 0, & \text{if } i \neq s, t, \\ -1, & \text{if } i = t; \end{cases}$$
$$x_{ij}^{\,v-1} \leq x_{ij} \leq c_{ij} - x_{ij}^{\,v-1},\ (i, j) \in U.$$

Actually MF(X_1) is a problem of the shortest augmenting path p in the incremental graph $G_I(X_{v-1})$ for the flow X_{v-1}. Hence, p is a PO solution (PO path) for the graph $G_I(X_{v-1})$ given the criteria $g_i(X)$, $i \in I_k$, i.e., p is a PO augmenting path for the flow X_{v-1}.

2. Let X_{v-1} be a PO solution of the problem MCF and p – a PO solution (path) in the incremental graph $G_I(X_{v-1})$. Let X_p be a flow with a value of 1 on the path p. We need to prove that the flow $X_v = X_{v-1} + X_p$ is a PO flow for the problem MCF.

The reverse is assumed – X_v is not a PO solution for MCF, i.e. there exists a flow Y_v for which the following is satisfied:

(3) $$g_i(Y_v) \leq g_i\,(X_v),\ \ g_{i1}(Y_v) < g_{i1}\,(X_v);\ i, i_1 \in I_k.$$

In conformance with the theorem for flow decomposition on paths and cycles, it can be written:

(4) $$Y_v = X_v + \sum_{i \in In_1} X_\sigma^{\,i} = X_{v-1} + X_p + \sum_{i \in In_1} X_\sigma^{\,i}$$

It is designed with $X_\sigma^{\,i}$, $i \in In_1$ flows on the cycles σ^i, $i \in In_1$ in graph G.

It follows from (3) and (4) that

$$\sum_{j\in In_1} g_i(X_\sigma^j) \le 0 \text{ and } \sum_{j\in In_1} g_{i1}(X_\sigma^j) < 0,\ i,\ i_1\in I_k.$$

It turns out, that at least for one of the flows

$$X_\sigma = X_{v-1} + \sum_{i\in In_1} X_\sigma^i \quad \text{and} \quad X_\sigma^p = X_p + \sum_{i\in In_1} X_\sigma^i$$

the conditions

$$g_i(X_\sigma) \le g_i(X_{v-1}),\ g_{i1}(X_\sigma) < g_{i1}(X_{v-1}),$$
$$g_i(X_p^\sigma) \le g_i(X_p),\ g_{i1}(X_p^\sigma) < g_{i1}(X_p)$$

are kept, which contradicts to the condition for Pareto optimality of both flows X_{v-1} and X_p.

The theorem proved describes the whole set of PO solutions of MCF problem.

3. Finding unsupported Pareto optimal solutions of the bicriteria network flow problem

For the bicriteria problem of a flow in a network (BCF), the image of the set of all PO solutions is a piecewise linear and convex set called efficient frontier (Ef). Its breakpoints are the images of its PO basic solutions and are called extreme points.

For the bicriteria integer problem of a flow in a network (BCIF), the question for determining the set of supported integer PO solutions (supported efficient flows) is solved [3,4]. It includes all PO basic solutions of the problem and the integer solutions along the segments, connecting each two neighbouring basic PO flows.

The images in the criteria space of the unsupported integer PO solutions of BCIF problem are found in the triangles, determined by each segment of Ef and the lines drawn across its breakpoints parallel to the axes in the criteria space (Fig. 1). The question of defining all unsupported integer PO solutions is not completely closed.

Let (X_1, X_2) and (Y_1, Y_2), where $X_i = g_i(X) = p_i$ and $Y_i = g_i(Y) = r_i$, $i = 1, 2$, be two neighbouring extreme points. Their corresponding points on the diagram in Fig. 1 are denoted by A and B. The flows X and Y, corresponding to these points are two neighbouring basic solutions in the space of MC*F* problem solutions. Let T_X and T_Y are their corresponding trees. Let T_Y be obtained by adding an appropriate arc (i_1, i_2) to T_X, sending an integer flow with a value of ε along the cycle σ, formed by T_X and (i_1, i_2) and excluding an appropriate arc (j_1, j_2) from this cycle.

Let $p_1 < r_1$ and $p_2 > r_2$. Let $q = p_2 - r_2 - 1$.

Let points A_i, $i=1,\dots, q$, be points from segment AB, for which:

$A_i = (g_1(Z_i), g_2(Z_i))$, Z_i – a PO flow for which $g_2(Z_i) = p_2 - i$, i.e. $g_2(Z_i)$ is integer valued.

Some of the points A_i are integer points that are received by sending flows with values 1, 2,…, $\varepsilon - 1$ along the cycle σ, i.e. they are supported PO solutions [3].

For the remaining points A_i, $i=1,\dots,q_1$, $q_1=q+1-\varepsilon$ (and all points from AB segment), their corresponding flows are obtained as convex linear combinations of the flows X and Y;

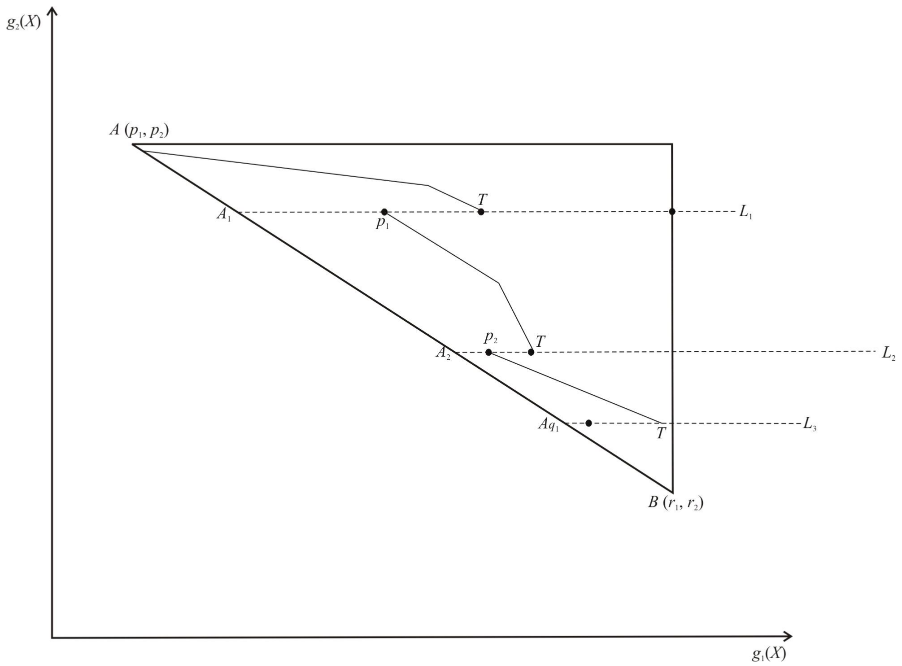

Fig.1

$$Z_i = \lambda_1 X + \lambda_2 Y = Y + Z_i(\sigma),\ g_2(Z_i) = g_2(Y) + g_2(Z_i(\sigma)).$$

Here $Z_i(\sigma)$ denotes a flow along the cycle σ, $\lambda_1 + \lambda_2 = 1$.

Let lines L_i, $i = 1, \ldots, q_1$, be drawn across each point A_i, parallel to the axis $g_1(X)$. Every unsupported PO solution of MCIF lies on any of L_i lines. There is no more than one unsupported PO solution on each line L_i. If the points P_1 and P_2 on the lines L_i and L_j, $i<j$, are corresponding to the two unsupported PO solutions, then the point P_2 lies in the right of P_1.

We design the flow X, $X_T = X$ as the first unsupported PO solution. An approach is offered for defining unsupported integer PO solutions. Its essence is that for each line L_i, $i=1,\ldots, q_1$, points T are successively found from this line, for which the corresponding flows can be defined. The flow in point T is defined, adding to the flow X_T flow Z_{ij} along a circulation σ_j, so that the costs $g_2(Z_i)$ and $g_2(X_T+Z_{ij})$ are equal and $g_1(Y) > g_1(X_T+Z_{ij}) > g_1(Z_i)$. A point T forms an interval with the point A_i, in which there is at least one integer point. Every such integer point is "checked" defining its corresponding flow X_I like convex linear combination of the flows on the points A_i and T. In case it is integer or the integer points of L_i are already considered, $X_T = X_{I.}$ The points along the next line are examined and so on.

In the opposite case we "enlarge" the interval (A_i, T) searching in similar way another point T on L_i in the right of the already investigated.

The formal content of the procedure suggested is as follows:

1. $i=1$, $j=1$, flow $X_i = X$.

The flow Z_i is determined for point A_i as a convex linear combination of the flows X and Y.

2. If Z_i is integer and $i = q_1$, end.

Otherwise if Z_i is integer, $i:=i+1$, go to 2.

In case Z_i is not integer, go to 4.

3. Let $G_I(X)$ be the incremental graph of G for the flow X.

4. The following problem is stated

$$\min\,(-g_1(X)), \tag{5}$$

$$X \in X(G_I(X_i),0), \tag{6}$$

$$g_2(X) \le 0.$$

The problem has got feasible solutions, among them being the points from line L_i. The search for an optimal solution is not necessary. An algorithm is applied, similar to the algorithm of the negative cycles for seeking a minimal flow for the problem defined by (5) and (6), the flow being altered only along these cycles for which $g_2(X_i) \le 0$.

6. Let at iteration j a circulation σ_j be found with costs $g_1(\sigma_j)$ and $g_2(\sigma_j)$ respectively, and with maximal feasible flow along it Z_{ij} with a value w_{ij}.

The value v_{ij} is determined as

$$v_{ij} = \min\,(w_{ij},(g_2(X) - g_2(Z_i)) \,/\, |\, g_2(\sigma_j)|\,).$$

If $(g_2(X) - g_2(Z_i)) \,/ |\, g_2(\sigma_j)|\,) \le w_{ij}$, the flow X_j is created,

$$X_j = X + v_{ij} / w_{ij}\, Z_{ij}\,.$$

Passing to 8.

Otherwise the flow X_i is formed:

$$X_i := X_j + Z_{ij}\,.$$

7. The incremental graph $G_I(X_j)$ is defined and Step 5 is executed.

8. If in the interval $[g_2(Z_i), g_2(X_j)]$, an integer number t is found, then the number λ is defined:

$$\lambda = (t - g_1(Z_j)) / \,(g_1(X_j) - g_1(Z_i)).$$

The flow $X_t = (1 - \lambda\,)Z_i + X_j$ is determined.

In case the flow X_t is integer and $i < q_1$, the flow X_t is unsupported PO solution. It is assigned $i:= i+1$, $X_I = X_t$ and Step 2 is completed.

If the flow X_t is not integer, Step 7 is executed.

4. Conclusion

Theorem 1 describes the whole set of PO solutions of MCF problem, but the application of an algorithm – an analogue to the algorithm for the minimal path in the single-criterion case, is practically impossible. The problem for a multicriteria PO path is NP-hard [2].Of course, some algorithms could be developed determining a part of the set of PO flows with the active participation of the decision maker.

The eventual case when for the point P_i, the convex linear combination of the flows in the points A_i and T, is not integer, but an integer flow exists, is still not investigated. It may be supposed that there is a circulation σ which consists of the arcs with non integer flow and $g_1(\sigma) = g_2(\sigma) = 0$.

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