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Some Properties of the Characteristic Polynomial of a Nonnegative Matrix*

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Abstract: *In this paper, we extend and generalize some of the well known spectral properties of a nonnegative matrix by establishing analogous properties for the zeros of derivatives of the characteristic polynomial of such a matrix.*

Keywords: *nonnegative matrix,Perron-Frobenius theory, characteristic polynomial.*

1. Introduction

Square matrices with nonnegative entries are an important class in matrix theory having a number of applications. The spectral theory of nonnegative matrices is known as Perron-Frobenius theory due to the fundamental works of O. Perron in 1907 and G. Frobenius in 1908-1912. The focal point in this theory is that the spectral radius of a nonnegative matrix A is an eigenvalue of A.

After Perron and Frobenius, many authors have proposed different proofs and a wide variety of extensions and generalizations of the basic theory. In most of the well known books, e.g. [1, 7, 9] the main theorems are stated and proved following H. Wilandt's approach based on variational characteristics of the spectral radius. This approach leads to effective bounds for the spectral radius in terms of the matrix entries and also, it can be easily adapted to more general linear-operator settings. Spectral properties extending the results of Perron and Frobenius have been obtained via topological methods following from the Brouwer fixed point theorem [6], by employing geometrical tools [5, 18] and by means of analytical arguments related to the Vivanti-Pringsheim theorem on power series with nonnegative coefficients [15]. The theory of nonnegative matrices has been generalized to the case of linear operators leaving invariant a cone in a partially ordered Banach space. There is an extensive literature in this direction including the notable contributions in [4, 10, 11, 19]. The main results in the case of an n-dimensional Euclidean space are presented in Ch. 1 of [3].

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In the recent literature, Perron-Frobenius theory continues to be a subject of intensive research. A simple and elegant spectral-theoretic proof of the fact that the spectral radius of a nonnegative matrix is necessarily an eigenvalue can be found in [20]. Also, one can encounter extensions to matrix polynomials [14], matrices with complex elements [17], nonlinear Perron-Frobenius theory, etc.

The most important applications of nonnegative matrices include iterative solution of large systems of linear equations and especially establishing convergence criteria for the iterative process, modeling and study of finite Markov chains, analysis of Leontief's input-output models in economics, solving linear complementarity problems in mathematical programming. All these topics are covered in Chapters 7-10 of [3]. Another interesting application of nonnegative matrices is in the area of information technologies and web-based information retrieval. References [2, 12, 16] describe matrix-theoretic techniques employed in the page ranking algorithms of the search engines Google and Ask Jeeves. Finally, it should be noted that nonnegative matrices are closely related with the systematic study of other matrix classes such as Z-matrices, M-matrices, etc.

In this paper, we study certain properties of the characteristic polynomial of a nonnegative matrix A. In particular, our aim is to examine which spectral properties of A are retained by the zeros of derivatives of the characteristic polynomial of A. As a result, we establish several such properties which enable us to extend and generalize well known results from Perron-Frobenius theory. Also, our work motivates an open problem in this area.

2. Results

Given a matrix $A=[a_{ij}] \in M_n(R)$, we shall denote by $f(\lambda)=\det(\lambda I - A)$ the characteristic polynomial of A and we shall write $A>0$ ($A\geq 0$) if all entries of A are positive (nonnegative). The adjoint of $\lambda I - A$ will be denoted by $B(\lambda) = [b_{ij}(\lambda)]$, $i, j = 1,\ldots,n$. Thus $B(\lambda)$ is a matrix polynomial with dimension n and degree $n-1$. We recall that the eigenvalues of A are the roots of the equation $f(\lambda) = 0$.

The basic spectral properties of a positive matrix are summarized in the following theorem [7, 9, 13].

Theorem 1. If $A \in M_n(R)$ and $A > 0$ then:

(i) $f(\lambda) = 0$ has at least one positive root;

(ii) the maximal positive root $\rho(A)$ of $f(\lambda) = 0$ is a simple root;

(iii) $B(\lambda) > 0$ for real $\lambda \geq \rho(A)$;

(iv) $\rho(A) > |\lambda|$ for each root $\lambda \neq \rho(A)$ of $f(\lambda)=0$;

(v) there is a positive eigenvector corresponding to $\rho(A)$, i.e. $Ax = \rho(A)x$, where $x \in R^n$, $x > 0$ is unique up to a scalar multiple.

The root $\rho(A)$ in Theorem 1 is often referred to as the Perron root of A. In the case of a nonnegative irreducible matrix, conditions (i)-(iii) and (v) also hold and condition (iv) is replaced by $\rho(A) \geq |\lambda|$ for each root λ of $f(\lambda) = 0$. If A is an arbitrary nonnegative matrix then, in general, weaker conditions hold, i. e. $f(\lambda) = 0$ has a nonnegative root $\rho(A)$ such that $B(\lambda) \geq 0$ for real $\lambda \geq \rho(A)$, $\rho(A) \geq |\lambda|$ for each root λ of $f(\lambda) = 0$ and there is a nonnegative eigenvector of A corresponding to $\rho(A)$. In this case, both the algebraic and geometrical multiplicity of $\rho(A)$ may be greater than one.

In the rest of this section, we shall extend conditions (i)-(iii) of Theorem 1 by including the derivatives of the characteristic polynomial of A. The following notation is used. Let $f^{(k)}(\lambda) = d^k f(\lambda)/d\lambda^k$ for $k = 0, 1, \ldots, n$, where $f^{(0)}(\lambda) = f(\lambda)$. The derivatives of $B(\lambda)$ are denoted by $d^kB(\lambda)/d\lambda^k=[b_{ij}^{(k)}(\lambda)]$, $i, j =1, 2, \ldots, n$, $k = 0, 1, \ldots, n-1$ with $d^0B(\lambda)/d\lambda^0 = B(\lambda)$. Let $N = \{1, 2, \ldots, n\}$. For a subset $\alpha \subset N$, the cardinality of α is denoted by $|\alpha|$ and the complement of α in N by $\overline{\alpha}$. Given $\alpha \subset N$ and $\beta \subset N$, $(\lambda I - A)(\alpha, \beta)$ will denote the submatrix of $\lambda I - A$ obtained by deleting rows and columns with indices in α and β, respectively. If $\alpha = \beta$, we shall write $(\lambda I - A)(\alpha)$ instead of $(\lambda I - A)(\alpha, \beta)$. Also, we shall use the notation $B^{(\alpha)}(\lambda) = \mathrm{adj}((\lambda I - A)(\alpha))$. Thus $B^{(\alpha)}(\lambda)$ is a matrix polynomial with dimension $n-|\alpha|$ and degree $n-|\alpha|-1$. The following determinant expansion will be used. Let $A=[a_{ij}]$, $i, j = 1, 2, \ldots, n$. For each i and j, $\det A = (-1)^{i+j}(a_{ij}\det A(\{i\},\{j\}) - a_{i(j)}\mathrm{adj}A(\{i\},\{j\})a_{(i)j})$, where $a_{i(j)} = A(\{i\},\overline{\{j\}})$ and $a_{(i)j} = A(\overline{\{i\}},\{j\})$.

We have the following result.

Theorem 2. Let $n \geq 2$, $A \in M_n(R)$ and $A > 0$. The following conditions hold for each $k = 0, 1, \ldots, n-2$:

(i) $f^{(k)}(\lambda) = 0$ has at least one positive root;

(ii) the maximal positive root $\rho^{(k)}$ of $f^{(k)}(\lambda) = 0$ is a simple root;

(iii) $d^kB(\lambda)/d\lambda^k > 0$ for real $\lambda \geq \rho^{(k)}$;

(iv) $\rho^{(0)} > \rho^{(1)} > \ldots > \rho^{(n-2)} > \dfrac{1}{n}\sum_{i=1}^{n} a_{ii}$.

P r o o f. We shall prove the theorem by induction on n. If $n = 2$, we have $k = 0$ and (i)-(iv) can be easily checked. Let $n \geq 3$. Assuming that conditions (i)-(iv) hold for matrices of dimension less than n, we shall prove them for an nxn matrix $A>0$.

Expanding $\det(\lambda I - A)$ by the i-th row and i-th column gives

(1) $$f^{(0)}(\lambda) = (\lambda - a_{ii})b_{ii}(\lambda) - a_{i(i)}B^{(\{i\})}(\lambda)a_{(i)i}$$

and by differentiating k times for $k = 1, 2, \ldots, n-2$ it is obtained

(2) $$f^{(k)}(\lambda) = kb_{ii}(\lambda) + (\lambda - a_{ii})b_{ii}^{(k)}(\lambda) - a_{i(i)}\frac{d^kB^{(\{i\})}(\lambda)}{d\lambda^k}a_{(i)i},$$

where $a_{i(i)} = A(\{i\},\overline{\{i\}})$ and $a_{(i)i} = A(\overline{\{i\}},\{i\})$. By the induction hypothesis, the polynomial $b_{ii}(\lambda)$ satisfies conditions (i)-(iv) for $k = 0, 1, \ldots, n-3$. It is easily seen that the $(n-2)$-nd derivative of $b_{ii}(\lambda)$ is a polynomial of degree one which has one positive zero and $d^{n-2}B^{(\{i\})}(\lambda)/d\lambda^{n-2}=(n-2)!I$. So, we shall denote by $\rho_{\{i\}}^{(k)}$ the maximal positive root of $b_{ii}^{(k)}(\lambda) = 0$ for $k = 0, 1, \ldots, n-2$. From (1), it follows that $f^{(0)}(\rho_{\{i\}}^{(0)}) < 0$ since $B^{(\{i\})}(\lambda) > 0$ for $\lambda \geq \rho_{\{i\}}^{(0)}$. Considering (2), we can write

$$b_{ii}^{(k-1)}(\lambda) = (\lambda - \rho_{\{i\}}^{(k-1)})\varphi_i(\lambda),$$

where $\varphi_i(\lambda)$ is a polynomial of degree $n-k-1$ and $\varphi_i(\rho_{\{i\}}^{(k-1)}) > 0$ due to the simplicity and maximality properties of $\rho_{\{i\}}^{(k-1)}$. On the other hand, since $\rho_{\{i\}}^{(k)}$ is the maximal positive zero of $b_{ii}^{(k)}(\lambda)$, it follows that $\varphi_i(\lambda)$ has no zeros in the interval $(\rho_{\{i\}}^{(k)}, \rho_{\{i\}}^{(k-1)}]$ and hence $\varphi_i(\rho_{\{i\}}^{(k)}) \geq 0$. Thus, we have

(3) $$b_{ii}^{(k-1)}(\rho_{\{i\}}^{(k)}) = (\rho_{\{i\}}^{(k)} - \rho_{\{i\}}^{(k-1)})\varphi_i(\rho_{\{i\}}^{(k)}) \le 0$$

and (2) and (3) imply that $f^{(k)}(\rho_{\{i\}}^{(k)}) < 0, k = 1, 2, ..., n-2$ since the last term in the right-hand side of (2) is negative for $\lambda \ge \rho_{\{i\}}^{(k)}$. Therefore,

(4) $$f^{(k)}(\rho_{\{i\}}^{(k)}) < 0, k = 0, 1, ..., n-2 .$$

As the coefficient of the highest degree term in $f^{(k)}(\lambda)$ is positive, it follows from (4) that $f^{(k)}(\lambda)=0$ has at least one positive root greater than $\rho_{\{i\}}^{(k)}$. Let $\rho^{(k)}$ denote the maximal positive root of $f^{(k)}(\lambda)=0$. Since (1) and (2) are valid for every i = 1, 2, …, n, it also follows that

(5) $$\rho^{(k)} > \max_{1\le i\le n} \rho_{\{i\}}^{(k)}, k = 0, 1, ..., n-2 .$$

Taking into account that $f^{(k)}(\lambda) = \sum_{i=1}^{n} b_{ii}^{(k-1)}(\lambda), k = 1, 2, ..., n$, inequality (5) shows that $\rho^{(k)}$ is a simple root of $f^{(k)}(\lambda) = 0$ for k = 0, 1, …, n–2 and $\rho^{(0)} > \rho^{(1)} > \ldots > \rho^{(n-2)} > \rho^{(n-1)}$, where $\rho^{(n-1)} = \frac{1}{n}\sum_{i=1}^{n} a_{ii}$ is the root of $f^{(n-1)}(\lambda)=0$. This proves conditions (i), (ii) and (iv).

In order to prove (iii), we note that inequality (5) also implies that the diagonal elements of $d^kB(\lambda)/d\lambda^k$ are positive for $\lambda \ge \rho^{(k)}$, k = 0, 1, …, n–2. Considering the off-diagonal elements, we can write $b_{ij}(\lambda)=(-1)^{i+j}\det(\lambda I-A)(\{j\},\{i\})$, i, j = 1, 2, …, n. If $i \ne j$, then expanding $\det(\lambda I-A)(\{j\},\{i\})$ by the row and column of $(\lambda I-A)(\{j\},\{i\})$ which contain the element $-a_{ij}$, it is obtained

(6) $$b_{ij}(\lambda) = (-1)^{i+j}(-1)^{i+j-1}(-a_{ij}\det(\lambda I - A)(\{i, j\}) - a_{i(i,j)}B^{(\{i,j\})}(\lambda)a_{(i,j)j}) =$$

$$= a_{ij}\det(\lambda I - A)(\{i, j\}) + a_{i(i,j)}B^{(\{i,j\})}(\lambda)a_{(i,j)j} ,$$

where $a_{i(i,j)} = A(\overline{\{i\}},\{i, j\})$ and $a_{(i,j)j} = A(\{i, j\},\overline{\{j\}})$. Since $\det(\lambda I-A)(\{i, j\})$ is the characteristic polynomial of a principal submatrix of A of dimension n–2, it is easily seen from (6) that $b_{ij}^{(n-2)}(\lambda) = (n-2)!a_{ij} > 0, i \ne j$. This proves (iii) for k=n–2 and we shall proceed with cases k = 0, 1, …, n–3. If $n \ge 4$, then by the induction hypothesis $\det(\lambda I-A)(\{i,j\})$ satisfies conditions (i)-(iv) for k = 0, 1, …, n – 4. Also, for $n \ge 3$, the (n–3)-rd derivative of $\det(\lambda I-A)(\{i, j\})$ is a polynomial of degree one which has one positive zero and $d^{n-3}B^{(\{i,j\})}(\lambda)/d\lambda^{n-3}=(n-3)!I$. So, denoting by $\rho_{\{i,j\}}^{(k)}$ the maximal positive zero of the k-th derivative of $\det(\lambda I-A)(\{i, j\})$ for $n \ge 3$ and k = 0, 1, …, n–3, it immediately follows from (6) that

(7) $$b_{ij}^{(k)}(\lambda) > 0 \text{ for } \lambda \ge \rho_{\{i,j\}}^{(k)}, k = 0, 1, ..., n-3, i \ne j .$$

Now, for each i = 1, 2, …, n, $b_{ii}(\lambda)$ is the characteristic polynomial of a principal submatrix of A of dimension n–1 and hence, an inequality analogous to (5) holds for the maximal positive zeros of the derivatives of $b_{ii}(\lambda)$, i.e.:

(8) $$\rho_{\{i\}}^{(k)} > \max_{1\le j\le n, j\ne i} \rho_{\{i,j\}}^{(k)}, k=0,1,\ldots,n-3,\ i=1,2,\ldots,n.$$

From (5), (7) and (8), it follows that $b_{ij}^{(k)}(\lambda) > 0$ for $\lambda \ge \rho^{(k)}$, $k = 0, 1, \ldots, n-3$ and $i \ne j$ which completes the proof.

A similar inductive proof can be constructed for nonnegative matrices. In this case, we have the following result.

Theorem 3. Let $A \in M_n(R)$ and $A \ge 0$. The following conditions hold for each $k = 0, 1, \ldots, n-1$:

(i) $f^{(k)}(\lambda)=0$ has at least one nonnegative root;

(ii) $d^kB(\lambda)/d\lambda^k \ge 0$ for real $\lambda \ge \rho^{(k)}$, where $\rho^{(k)}$ is the maximal nonnegative root of $f^{(k)}(\lambda)=0$;

(iii) $\rho^{(0)} \ge \rho^{(1)} \ge \ldots \ge \rho^{(n-1)} = \frac{1}{n}\sum_{i=1}^{n} a_{ii}$.

An obvious consequence of condition (iii) of Theorem 3 is that if $A \in M_n(R)$ and $A \ge 0$ with tr$A>0$ then $\rho^{(k)}$ is strictly positive for each $k = 0, 1, \ldots, n-1$.

The next corollaries provide bounds for $\rho^{(k)}$ in terms of the entries of A. We shall use the following more general notation. Given an $n\times n$ matrix $A>0$ ($A\ge 0$) and any $\alpha \subset N$, $0 \le |\alpha| \le n-1$, $\rho_{\alpha}^{(k)}$ will denote the maximal positive (nonnegative) zero of the k-th derivative of $\det(\lambda I-A)(\alpha)$ for $k = 0, 1, \ldots, n-|\alpha|-1$. As in Theorems 2 and 3, we shall write $\rho^{(k)}$ instead of $\rho_{\otimes}^{(k)}$ in the cases $\alpha = \otimes$ and $k = 0, 1, \ldots, n-1$.

Corollary 1. The maximal positive root $\rho^{(k)}$ in Theorem 2 satisfies

(9) $$\rho^{(k)} > \frac{1}{k+1} \max_{\alpha\subset N, |\alpha|=k+1} \sum_{i\in\alpha} a_{ii}, k=0,1,\ldots,n-2.$$

Proof. A consecutive application of inequality (5) to $\det(\lambda I-A)(\alpha)$ for each $k = 0, 1, \ldots, n-2$ and $1 \le |\alpha| \le n-k-1$ gives

(10) $$\rho^{(k)} > \max\{\rho_{\alpha}^{(k)} : \alpha \subset N, 1 \le |\alpha| \le n-k-1\}, k=0,1,\ldots,n-2$$

and the special case of (10)

(11) $$\rho^{(k)} > \max\{\rho_{\alpha}^{(k)} : \alpha \subset N, |\alpha| = n-k-1\}, k=0,1,\ldots,n-2$$

yields (9).

It can be easily seen that in the case of a nonnegative matrix A, the inequality in (9) is not strict, i.e. the nonnegative root $\rho^{(k)}$ in Theorem 3 satisfies

(12) $$\rho^{(k)} \ge \frac{1}{k+1} \max_{\alpha\subseteq N, |\alpha|=k+1} \sum_{i\in\alpha} a_{ii}, k=0,1,\ldots,\ n-1.$$

The next corollary is a consequence of Theorem 3 and the fact that the Perron root of a nonnegative matrix A is greater or equal to the minimal row sum of A and less or equal to the maximal row sum of A. We shall also use the following simple property of the characteristic polynomial $f(\lambda)$ of an $n\times n$ nonnegative matrix A.

The n-th derivative of $f(\lambda)$ is $f^{(n)}(\lambda)=n!>0$ and in this case we formally set $\rho^{(n)}=-\infty$. Thus for real values of λ, $f^{(n-1)}(\lambda)$ is a linear strictly increasing function of λ in the interval $(\rho^{(n)},\infty)$ and $f^{(n-1)}(\lambda)>0$ for $\lambda>\rho^{(n-1)}$. This property of $f^{(n-1)}(\lambda)$ implies that $f^{(n-2)}(\lambda)$ is a strictly increasing function of λ in the interval $(\rho^{(n-1)},\infty)$ and since $\rho^{(n-2)}\geq\rho^{(n-1)}$, we have $f^{(n-2)}(\lambda)>0$ for $\lambda>\rho^{(n-2)}$. Continuing by induction, it is seen that $f^{(k)}(\lambda)$ is a strictly increasing function of λ in the interval $(\rho^{(k+1)},\infty)$ and

(13) $$f^{(k)}(\lambda)>0 \text{ for } \lambda>\rho^{(k)},\ k = 0, 1, \dots, n-1.$$

By applying inequality (13) to the diagonal elements $b_{ii}(\lambda)$ of $B(\lambda)$ and taking into account that $\rho^{(k)}\geq\max_{1\leq i\leq n}\rho^{(k)}_{\{i\}},\ k=0,1,\dots,n-1$, condition (ii) of Theorem 3 can be augmented by the inequalities

(14) $$b_{ii}^{(k)}(\lambda)>0 \text{ for real } \lambda>\rho^{(k)},\ k=0,1,\dots,n-1,\ i=1,2,\dots,n.$$

Corollary 2. The maximal nonnegative root $\rho^{(k)}$ in Theorem 3 satisfies

(15) $$\min_{\alpha\subset N,|\alpha|=k}\ \min_{i\in\overline{\alpha}}\sum_{j\in\overline{\alpha}}a_{ij}\leq\rho^{(k)}\leq\max_{\alpha\subset N,|\alpha|=k}\ \max_{i\in\overline{\alpha}}\sum_{j\in\overline{\alpha}}a_{ij},\ k=0,1,\dots,n-1.$$

P r o o f. Inequality (15) is trivial for n–1. Also, for k=0 inequalities (15) are well known and represent the above mentioned property of the Perron root of A. Let $n\geq2$ and consider the cases $k = 1, 2, \dots, n-1$. First, we shall prove that

(16) $$\min_{1\leq i\leq n}\rho^{(k-1)}_{\{i\}}\leq\rho^{(k)}\leq\max_{1\leq i\leq n}\rho^{(k-1)}_{\{i\}},\ k=1,2,\dots,n-1.$$

Since

(17) $$f^{(k)}(\rho^{(k)})=\sum_{i=1}^{n}b_{ii}^{(k-1)}(\rho^{(k)})=0,\ k=1,2,\dots,n-1$$

it follows that for each $k = 1, 2, \dots, n-1$ there is at least one index $i_k\in N$ such that

(18) $$b_{i_k i_k}^{(k-1)}(\rho^{(k)})\geq0$$

and at least one index $j_k\in N$, $j_k\neq i_k$, such that

(19) $$b_{j_k j_k}^{(k-1)}(\rho^{(k)})\leq0.$$

As $b_{j_k j_k}^{(k-1)}(\lambda)\to\infty$ for real $\lambda\to\infty$, inequality (19) shows that the maximal nonnegative zero $\rho^{(k-1)}_{\{j_k\}}$ of $b_{j_k j_k}^{(k-1)}(\lambda)$ satisfies $\rho^{(k-1)}_{\{j_k\}}\geq\rho^{(k)}$. On the other hand, it follows from (14) that $b_{i_k i_k}^{(k-1)}(\lambda)$ is a strictly increasing function in the interval $(\rho^{(k)},\infty)$ and thus inequality (18) shows that the maximal nonnegative zero $\rho^{(k-1)}_{\{i_k\}}$ of $b_{i_k i_k}^{(k-1)}(\lambda)$ satisfies $\rho^{(k-1)}_{\{i_k\}}\leq\rho^{(k)}$. This proves (16).

Now, since inequalities (16) also hold for every principal submatrix of A, it is easily seen that for each $k = 1, 2, \dots, n-1$ there are subsets of N $\alpha_1 \subset \alpha_2 \subset \dots \subset \alpha_k$ and $\beta_1 \subset \beta_2 \subset \dots \subset \beta_k$, $|\alpha_i| = |\beta_i| = i$, $i = 1, 2, \dots, k$ such that

(20) $$\rho_{\alpha_k}^{(0)} \le \dots \le \rho_{\alpha_1}^{(k-1)} \le \rho^{(k)} \le \rho_{\beta_1}^{(k-1)} \le \dots \le \rho_{\beta_k}^{(0)} .$$

In (20), $\rho_{\alpha_k}^{(0)}$ is the Perron root of $A(\alpha_k)$ and satisfies

(21) $$\rho_{\alpha_k}^{(0)} \ge \min_{i \in \alpha_k} \sum_{j \in \alpha_k} a_{ij} .$$

Similarly, $\rho_{\beta_k}^{(0)}$ is the Perron root of $A(\beta_k)$ and satisfies

(22) $$\rho_{\beta_k}^{(0)} \le \max_{i \in \beta_k} \sum_{j \in \beta_k} a_{ij} .$$

Thus, inequalities (15) follow from (20), (21) and (22).

We note that in the special case $k = n - 1$, (16) and (15) respectively yield the trivial inequalities

$$\frac{1}{n-1} \min_{1 \le j \le n} \sum_{i \ne j} a_{ii} \le \frac{1}{n} \sum_{i=1}^{n} a_{ii} \le \frac{1}{n-1} \max_{1 \le j \le n} \sum_{i \ne j} a_{ii}$$

and

$$\min_{1 \le i \le n} a_{ii} \le \frac{1}{n} \sum_{i=1}^{n} a_{ii} \le \max_{1 \le i \le n} a_{ii} .$$

It is easily seen that Corollary 2 can be generalized by noting that matrices A, $S^{-1}AS$ and $S^{-1}A^{T}S$ have the same characteristic polynomial for any nonsingular matrix S. If $A \ge 0$ and $S = \mathrm{diag}(x_1, x_2, \dots, x_n)$ with $x_i > 0$, $i = 1, 2, \dots, n$ then applying Corollary 2 to $S^{-1}AS \ge 0$ and $S^{-1}A^{T}S \ge 0$ gives the following result.

For each positive vector $x \in R^n$, the maximal nonnegative root $\rho^{(k)}$ in Theorem 3 satisfies

(23) $$\min_{\alpha \subset N, |\alpha| = k} \min_{i \in \alpha} \frac{1}{x_i} \sum_{j \in \alpha} a_{ij} x_j \le \rho^{(k)} \le \max_{\alpha \subset N, |\alpha| = k} \max_{i \in \alpha} \frac{1}{x_i} \sum_{j \in \alpha} a_{ij} x_j , \; k = 0, 1, \dots, n-1$$

and

(24) $$\min_{\alpha \subset N, |\alpha| = k} \min_{j \in \alpha} x_j \sum_{i \in \alpha} \frac{a_{ij}}{x_i} \le \rho^{(k)} \le \max_{\alpha \subset N, |\alpha| = k} \max_{j \in \alpha} x_j \sum_{i \in \alpha} \frac{a_{ij}}{x_i} , \; k = 0, 1, \dots, n-1 .$$

3. Comments and concluding remarks

Theorems 2 and 3 in the special case $k = 0$ represent well known results from Perron-Frobenius theory concerning the spectral properties of a nonnegative matrix. For $k > 0$, we have extended these results by showing that analogous properties are also valid for derivatives of the characteristic polynomial of such a matrix. As immediate consequences, Corollaries 1 and 2 provide bounds for the maximal nonnegative zeros of the characteristic polynomial and its derivatives which include the corresponding bounds for the spectral radius of a nonnegative matrix. A variational type characterizations of these zeros are also given by inequalities (23) and (24).

The results in the previous section rise the following question. Is the maximal nonnegative zero of the k-th derivative of the characteristic polynomial of an nonnegative matrix greater or equal to the absolute value of any other zero of this derivative for each $k = 1, 2, \ldots, n-1$. An affirmative answer follows easily from Theorems 2 and 3 when $n \leq 3$ and also, in some special cases of nonnegative matrices with arbitrary dimension. In general however, this question remains an open problem which needs further analysis.

As a final remark, it should be noted that the proof of Theorem 2 can be viewed as an extension of the inductive argument given by G. Frobenius in his proof of the theorem of O. Perron. The original works of Frobenius on nonnegative matrices from years 1908-1912 can not be readily found today but a proof of Perron's theorem attributed to Frobenius is given in the book of Gantmacher and Krein [8]. This proof came to the attention of the author of the present paper after he had proved Theorem 2.

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