BULGARIAN ACADEMY OF SCIENCES

CYBERNETICS AND INFORMATION TECHNOLOGIES • Volume **6**, No 2

Sofia • 2006

Reference-Neighborhood Scalarizing Problems of Multicriteria Integer Optimization

Krassimira Genova

Institute of Information Technologies, 1113 Sofia

Abstract: *The purpose of this paper is to propose reference-neighborhood scalarizing problems for finding (weak) Pareto optimal solutions of multicriteria optimization problems. The decision maker (DM) provides information about his/her preferences for choice of new Pareto optimal solution with respect to the criteria values at the current solution. The current solution and the DM's local preferences set a reference-neighborhood in the Pareto optimal set of the multicriteria problem solution, and the scalarizing problems search for a new (weak) Pareto optimal solution in this area.*

Keywords*: scalarizing problems, multicriteria integer optimization.*

1. Introduction

Several criteria (objective functions) are simultaneously optimized in the feasible set of solutions (alternatives) in the multicriteria optimization problems. In the general case a single solution, which optimizes the criteria, does not exist. However, there is a set of solutions in the variables' space and a respective set in the criteria space, which is characterized by the following: each improvement in the value of one criterion leads to deterioration in the value of at least one other criterion. These sets are called Pareto optimal sets. Every element of these sets could be a solution of the multicriteria optimization problem. In order to select a particular element, the so-called decision maker (DM) has to provide additional information. The information, which the DM sets, reflects his/her global preferences with respect to the quality of the solution obtained.

The scalarizing approach is one of the main approaches in solving multicriteria optimization problems. The basic representatives of the scalarizing approach (Benayoun et al. [1], Wierzbicki [14], Nakayama, Sawaragi [6], Steuer [9], Narula, Vassilev [7], Korhonen [4], Buchanan [2], Miett i n e n [5], V a s s i l e v a [11], V a s s i l e v et al. [10], V a s s i l e v a [12]), are the interactive algorithms. In the general case each interactive algorithm consists of two procedures – an optimization one and an evaluating one. These procedures are cyclically repeated until the stopping conditions are satisfied. During the evaluating procedure the DM estimates the obtained current Pareto optimal solution and either accepts it as the final (the most preferred) one, or sets his/her preferences in the search for a new solution. On the basis of these preferences a scalarizing problem is formed and subsequently solved in the optimization procedure. As a result a new Pareto optimal solution is obtained, which is presented to the DM for evaluation. The main feature of each scalarizing problem is that every optimal solution is a Pareto optimal solution of the corresponding multicriteria optimization problem. The scalarizing problem is a single-criterion optimization problem, which allows the application of the theory and methods of single-criterion optimization. A number of scalarizing problems and a set of interactive algorithms developed on their basis have been proposed so far. The different algorithms offer different possibilities to the DM for controlling or stopping the process of the final solution finding. On its hand, this searching process can be divided into two phases. In the first phase (the learning phase) the DM usually defines the region, in which he/she expects to find the most preferred solution, whereas in the second phase (the concluding phase), he/she is looking for this solution namely in this region. The interactive algorithms are especially appropriate for solving linear multicriteria optimization problems, in which the time for scalarizing problems solution (the time for a new solution expecting) does not play an important role.

The present paper describes reference-neighborhood scalarizing problems. The designation of the scalarizing problems is based on the region, in which a new Pareto optimal solution is sought. This region is defined by the current solution obtained and the preferences set by the DM. The reference-neighborhood summarizes the reference direction that is utilized in the scalarizing problems suggested by K o r h o-n e n [4]. The reference direction is set by the obtained current solution and the reference point. The components of the reference point equalize the preset desired (aspiration) levels of the criteria by the DM. The reference-neighborhood is defined by the obtained current solution, the preset desired (aspiration) levels by the DM and the desired directions of the alteration of the criteria values. The reference-neighborhood scalarizing problems are in the same group as the classification based scalarizing problems (M i e t t i n e n [5]), because the criteria in the reference-neighborhood scalarizing problems can be classified in different groups based on the DM's preferences. However, the classification of the criteria is not the defining factor for the designation of the proposed scalarizing problems, but the defined region, in which the new Pareto optimal solution is sought. The reference-neighborhood scalarizing problems are especially appropriate for solving multicriteria integer optimization problems because of their main features.

2. Problem formulation

The proposed reference-neighborhood scalarizing problem is designed for solving multicriteria linear integer problems (MLIP). These multicriteria problems can be formulated as

(1) $$\text{“max”}\{f_k(x), k \in K\}$$

subject to:

(2) $$\sum_{j \in N} a_{ij} x_j \leq b_i, \; i \in M,$$

(3) $$0 \leq x_j \leq d_j, \; j \in N,$$

(4) $$x_j \; - \; \text{integer}, \; j \in N,$$

where $f_k(x)$, $k \in K$, are linear criteria (objective functions); $f_k(x) = \sum_{j \in N} c_j^k x_j$ and symbol "max" means that all criteria are to be simultaneously maximized; $K = \{1, 2,..., p\}$, $M = \{1, 2,...,m\}$, $N = \{1, 2,...,n\}$ denote the index sets of the criteria, the linear constraints, and the decision variables, respectively: $x = (x_1, x_2,..., x_j,..., x_n)^{\mathrm{T}}$ is the vector of the decision variables.

The constraints (2)-(4) define the feasible region X_1 for the integer variables.

The problem (1)-(3) is a multicriteria linear programming problem (MLP). The feasible region for the continuous variables is denoted by X_2. Problem MLP is a relaxation of MLIP.

For clarity of the exposition, a few definitions of the used terms are given.

Definition 1. The solution x is called an efficient solution of MLP or MLIP, if there does not exist any other solution $\bar{x}$, such that the following inequalities are satisfied:

$$f_k(\bar{x}) \geq f_k(x) \text{ for every } k \in K \text{ and}$$
$$f_k(\bar{x}) > f_k(x) \text{ for at least one index.}$$

Definition 2. The solution x is called a weak efficient solution of MLP or MLIP, if there does not exist another solution $\bar{x}$ such that the following inequalities hold:

$$f_k(\bar{x}) > f_k(x), \text{for every } k \in K.$$

Definition 3. The solution x is called a (weak) efficient solution of MLP or MLIP, if $\bar{x}$ is either an efficient or a weak efficient solution.

Definition 4. The vector $f(x) = (f_1(x), ..., f_p(x))^{\mathrm{T}}$ is called a (weak) Pareto optimal solution in the criteria space, if x is a (weak) efficient solution in the variable space.

Definition 5. The vector $f^* = (f_1^*,..., f_p^*)^{\mathrm{T}}$ is called an ideal solution in the criteria (objective) space, if its every component f_k^* is derived as individual optimization of each criterion (objective function) $f_k(x)$ in the feasible space of MLP or MLIP.

Definition 6. A current preferred solution of MLP or MLIP is a (weak) Pareto optimal solution chosen by the DM at the current iteration. The most preferred solution of MLP or MLIP is the solution that satisfies the DM to the greatest degree.

3. Scalarizing problems

When solving a MLIP problem, the DM estimates and compares the currently obtained (weak) Pareto optimal solutions. If the DM looks for a better solution, he/she needs to set his/her preferences for the desirable or reasonable alterations of the values of some or all criteria. Depending on these preferences, the set of the criteria at each iteration can be indirectly divided into four or less than four classes, denoted as follows $K^{>}, K^{\geq}, K^{<}, K^{\leq}$. Each criterion $f_k(x),\ k \in K$, may belong to one of these classes, as given below:

$k \in K^{>}$, if the DM wishes the criterion $f_k(x)$ to be improved;

$k \in K^{\geq}$, if the DM wishes the criterion $f_k(x)$ to be improved by any desired (aspiration) value Δ_k;

$k \in K^{<}$, if the DM assumes the criterion $f_k(x)$ to be worsened;

$k \in K^{\leq}$, if the DM assumes the value of the criterion $f_k(x)$ to be deteriorated by no more than δ_k.

In order to obtain a (weak) Pareto optimal solution of MLIP problem, on the basis of the implicit criteria classification, done by the DM, the following scalarizing problems RNS1 {Reference-Neighborhood Scalarizing Problems) is proposed bellow.

To minimize

(5) $S(x)=$

$$=\max\left(\max_{k\in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x)}{\left|\bar{f}_k - f_k\right|}\right), \max_{k\in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|}\right), \max_{k\in K^{>}}\left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|}\right), \max_{k\in K^{<}}\left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|}\right)\right)$$

under the constraints:

(6) $$f_k(x) \geq f_k,\ k \in K^{>},$$

(7) $$f_k(x) \geq \tilde{f}_k,\ k \in K^{\leq},$$

(8) $$x \in X_1,$$

where f_k is the value of the criterion $f_k(x)$ in the current preferred solution, f_k^* is an ideal solution vector in the criteria space of MLIP, $\bar{f}_k = f_k + \Delta_k$ is the desired (aspiration) level of the criterion with an index $k \in K^{\geq}$, $\tilde{f}_k = f_k - \delta_k$, the DM agrees with worsening by value δ_k of the current value of the criterion with an index $k \in K^{\leq}$.

To obtain a (weak) Pareto optimal solution for MLP problem in the reference-neighborhood of the current preferred solution, we may use the scalarizing problem RNS1-L, which is obtained from RNS1 by replacing constraint (8) by constraint

(9) $$x \in X_2.$$

Theorem 1. The optimal solution of the scalarizing problem RNS1 is a weak efficient (Pareto optimal) solution of the multicriteria linear integer programming problem (1)-(4).

P r o o f. If the DM is looking for a new solution, which is better than the current one at least in one criterion, he/she needs to solve the scalarizing problem RNS1. Therefore we have to assume that $K^{\geq} \neq \varnothing$ or $K^{>} \neq \varnothing$.

Let x^* be an optimal solution of problem RNS1. Then the following condition is satisfied:

(10) $$S(x^*) \leq S(x), \quad x \in X_1,$$

and $f_k(x^*) \geq f_k,\ k \in K^{>};\ \ f_k(x^*) \geq \tilde{f}_k,\ k \in K^{\leq}.$

Let us assume that x^* is not a weak efficient solution of the initial MLIP (1)-(4). In this case there must exist another $x' \in X$, which is a weak efficient solution of MLIP (1)-(4). From the definition of weak efficient solution of MLIP and the inequalities (10), it follows that:

(11) $$f_k(x') > f_k(x^*),\ k \in K,$$

and $f_k(x') \geq f_k,\ k \in K^{>};\ \ f_k(x') \geq \tilde{f}_k,\ k \in K^{\leq}.$

After transformation of the objective function $S(x)$ of the scalarizing problem RNS1, using the inequalities (11), the following relation is obtained:

(12) $S(x') =$

$$= \max\left(\max_{k\in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x')}{\left|\bar{f}_k - f_k\right|} \right), \max_{k\in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x')}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k\in K^{>}}\left(\frac{f_k^* - f_k(x')}{\left|f_k^* - f_k\right|} \right), \max_{k\in K^{<}}\left(\frac{f_k^* - f_k(x')}{\left|f_k^* - f_k\right|} \right) \right) =$$

$$= \max\left(\max_{k\in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x^*)}{\left|\bar{f}_k - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|\bar{f}_k - f_k\right|} \right), \max_{k\in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x^*)}{\left|\tilde{f}_k - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|\tilde{f}_k - f_k\right|} \right), \right.$$

$$\left. \max_{k\in K^{>}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|f_k^* - f_k\right|} \right), \max_{k\in K^{<}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|f_k^* - f_k\right|} \right) \right) <$$

$$< \max\left(\max_{k\in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x^*)}{\left|\bar{f}_k - f_k\right|} \right), \max_{k\in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x^*)}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k\in K^{>}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} \right), \max_{k\in K^{<}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} \right) \right) =$$

$$= S(x^*)$$

It follows from (12) that $S(x') < S(x^*)$ and $f_k(x^*) \geq f_k,\ k \in K^{>},\ f_k(x^*) \geq \tilde{f}_k,$ $k \in K^{\leq}$, which contradicts to (10). Hence x^* is a weak efficient solution and $f(x^*)$ is a weak Pareto optimal solution in the criteria space of MLIP (1)-(4).

Consequence. Theorem 1 is true for arbitrary values of $f_k,\ k \in K$.

This consequence follows from the fact that the proof of Theorem 1 does not assume any conditions concerning the values of the criteria $f_k,\ k \in K$.

The current preferred solution of the multicriteria problem is a feasible solution of the current scalarizing problem RNS1, i.e., the scalarizing problem RNS1 has an initial feasible solution. This is a very important property, because the finding of a feasible solution of integer problems is an NP-problem. Furthermore, the feasible solutions of the scalarizing problem RNS1 are located near the Pareto optimal surface of the multicriteria problem in the criteria space. They belong to the reference-neighborhood space defined by the DM's preferences. According to the manner, which the DM uses to set up his preferences for alteration of the criteria values, the reference-neighborhood space could be very narrow or it could widen considerably, if the DM has set freely improvement or he/she has admitted free worsening for most of the criteria.

The solution of problem RNS1 is a weak Pareto optimal solution. A guarantee for obtaining a Pareto optimal solution, the problem RNS1 can be modified to problem RNS2, as follows below.

To minimize

(13) $T(x) =$

$$= \max\left(\max_{k\in K^{\geq}} \left(\frac{\bar{f}_k - f_k(x)}{\left|\bar{f}_k - f_k\right|} \right), \max_{k\in K^{\leq}} \left(\frac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k\in K^{>}} \left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|} \right), \max_{k\in K^{<}} \left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|} \right) \right) +$$

$$+ \delta \left(\sum_{k\in K^{\geq}} \left(\bar{f}_k - f_k(x)\right) + \sum_{k\in K^{\leq}} \left(\tilde{f}_k - f_k(x)\right) + \sum_{k\in K^{>}} \left(f_k - f_k(x)\right) + \sum_{k\in K^{<}} \left(f_k - f_k(x)\right) \right)$$

under the constraints:

(14) $$f_k(x) \geq f_k,\ k \in K^{>},$$

(15) $$f_k(x) \geq \tilde{f}_k,\ k \in K^{\leq},$$

(16) $$x \in X_1,$$

where δ is arbitrary small number.

Theorem 2. The optimal solution of the scalarizing problem RNS2 is an efficient (Pareto optimal) solution of the multicriteria linear integer programming problem (1)-(4).

P r o o f. If the DM is looking for a new solution, which is better than the current one at least in one criterion, he/she needs to solve the scalarizing problem RNS2. Therefore we have to assume that $K^{\geq} \neq \varnothing$ or $K^{>} \neq \varnothing$.

Let x^* be an optimal solution of the problem RNS2. Then the following conditions are satisfied:

(17) $$T(x^*) \leq T(x), \quad x \in X_1,$$

and $f_k(x^*) \geq f_k,\ k \in K^{>}$; $f_k(x^*) \geq \tilde{f}_k,\ k \in K^{\leq}$.

Let us assume that $x^* \in X_1$ is not an efficient (Pareto optimal) solution of the initial MLIP (1)-(4). In this case there must exist another $x^{'} \in X$, which is an efficient (Pareto optimal) solution of MLIP (1)-(4) and for which from the definition and according to inequalities (17), the following conditions are satisfied:

(18) $$f_k(x^{'}) \geq f_k(x^*),\ k \in K,$$

$$f_k(x^{'}) > f_k(x^*) \text{ for at least one index } l \neq k,$$

and $f_k(x') \geq \bar{f}_k,\ k \in K^{>}$; $f_k(x') \geq \tilde{f}_k,\ k \in K^{\leq}$.

After transformation of the objective function $T(x)$ of the scalarizing problem RNS2, using the inequalities (18), the following relation is obtained:

(19) $T(x') =$

$$= \max\left(\max_{k \in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x')}{\left|\bar{f}_k - f_k\right|} \right), \max_{k \in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x')}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k \in K^{>}}\left(\frac{f_k^* - f_k(x')}{\left|f_k^* - f_k\right|} \right), \max_{k \in K^{<}}\left(\frac{f_k^* - f_k(x')}{\left|f_k^* - f_k\right|} \right) \right) +$$

$$+ \delta\left(\sum_{k \in K^{\geq}} \left(\bar{f}_k - f_k(x')\right) + \sum_{k \in K^{\leq}} \left(\tilde{f}_k - f_k(x')\right) + \sum_{k \in K^{<} \cup K^{>}} \left(f_k - f_k(x')\right) \right) =$$

$$= \max\left(\max_{k \in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x^*)}{\left|\bar{f}_k - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|\bar{f}_k - f_k\right|} \right), \max_{k \in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x^*)}{\left|\tilde{f}_k - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|\tilde{f}_k - f_k\right|} \right), \right.$$

$$\left. \max_{k \in K^{>}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|f_k^* - f_k\right|} \right), \max_{k \in K^{<}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} + \frac{f_k(x^*) - f_k(x')}{\left|f_k^* - f_k\right|} \right) \right) +$$

$$+ \delta\left(\sum_{k \in K^{\geq}} \left(\left(\bar{f}_k - f_k(x^*)\right) + \left(f_k(x^*) - f_k(x')\right) \right) + \sum_{k \in K^{\leq}} \left(\left(\tilde{f}_k - f_k(x^*)\right) + \left(f_k(x^*) - f_k(x')\right) \right) + \right.$$

$$\left. + \sum_{k \in K^{<} \cup K^{>}} \left(\left(f_k - f_k(x^*)\right) + \left(f_k(x^*) - f_k(x')\right) \right) \right) <$$

$$< \max\left(\max_{k \in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x^*)}{\left|\bar{f}_k - f_k\right|} \right), \max_{k \in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x^*)}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k \in K^{>}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} \right), \max_{k \in K^{<}}\left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} \right) \right) +$$

$$+ \delta\left(\sum_{k \in K^{\geq}} \left(\bar{f}_k - f_k(x^*)\right) + \sum_{k \in K^{\leq}} \left(\tilde{f}_k - f_k(x^*)\right) + \sum_{k \in K^{<} \cup K^{>}} \left(f_k - f_k(x^*)\right) \right) = T(x^*).$$

It follows from (19) that $T(x^{'}) < T(x^*)$ and $f_k(x^*) \geq \bar{f}_k,\ k \in K^{>}$; $f_k(x^*) \geq \tilde{f}_k$, $k \in K^{\leq}$, which contradicts to (17). Hence x^* is an efficient solution and $f(x^*)$ is a Pareto optimal solution in the criteria space of MLIP (1)-(4).

In order to find a Pareto optimal solution of problem MLP, we may use the scalarizing problem RNS2 by replacing constraint (16) by constraint (9). We denote the obtained relaxed problem by RNS2-L.

Because the objective function of the scalarizing problems RNS1 and RNS2 is nondifferentiable, each one of them could be converted into an equivalent optimization problem by adding additional variables and limits, but with a differential objective function. N e m h a u s e r, W o l s e y [8], W o l s e y [15]. The equivalent mixed integer programming problem of problem RNS1, denoted by RNS1e, can be presented as follows:

(20) $$\alpha \geq \frac{\bar{f}_k - f_k(x)}{\left|\bar{f}_k - f_k\right|},\ k \in K^{\geq}$$

under the constraints:

(21) $$\alpha \geq \frac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|},\ k \in K^{\leq},$$

(22) $$\alpha \geq \frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|},\ k \in K^{<},$$

(23) $$\alpha \geq \frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|},\ k \in K^{>},$$

(24) $$f_k(x) \geq f_k,\ k \in K^{>},$$

(25) $$f_k(x) \geq \tilde{f}_k,\ k \in K^{\leq},$$

(26) $$f_k(x) \geq \tilde{f}_k,\ k \in K^{\leq},$$

(27) $$x \in X_1,$$

(28) $$\alpha - \text{arbitrary}.$$

Problems RNS1 and RNS1e have the same feasible sets of variables. The values of their objective functions are also equal. This follows from the following assertion.

Theorem 3. The optimal values of the objective functions of scalarizing problems RNS1 and RNS1e are equal, i.e.,

$$\min_{x \in X_1}(\alpha) = \min_{x \in X_1}\left(\max\left(\max_{k \in K^{\geq}}\left(\frac{\bar{f}_k - f_k(x)}{\left|\bar{f}_k - f_k\right|} \right), \max_{k \in K^{\leq}}\left(\frac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k \in K^{>} \cup K^{<}}\left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|} \right) \right) \right).$$

P r o o f. It follows from (21) that $\alpha \geq \frac{\bar{f}_k - f_k(x)}{\left|\bar{f}_k - f_k\right|},\ k \in K^{\geq}$. Since this inequality is true for every $k \in K^{\geq}$, it is also true that

(29) $$\alpha \geq \max_{k \in K^{\geq}} \frac{\bar{f}_k - f_k(x)}{\left|\bar{f}_k - f_k\right|} .$$

It follows from (22) that $\alpha \geq \dfrac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|}, k \in K^{\leq}$. Since this inequality is true for every $k \in K^{\leq}$, it is also true that

(30) $$\alpha \geq \max_{k \in K^{\leq}} \frac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|} .$$

Similarly to (23), it follows that

(31) $$\alpha \geq \max_{k \in K^{<}} \left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|} \right),$$

and based on (24), that

(32) $$\alpha \geq \max_{k \in K^{>}} \left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|} \right).$$

From (29), (30), (31) and (32) it can be written:

(33) $$\alpha \geq \max \left(\max_{k \in K^{\geq}} \left(\frac{\bar{f}_k - f_k(x)}{\left|\bar{f}_k - f_k\right|} \right), \max_{k \in K^{\leq}} \left(\frac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k \in K^{>}} \left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|} \right), \max_{k \in K^{<}} \left(\frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|} \right) \right) .$$

Let x^* be an optimal solution of problem RNS1e. Then:

(34) $$\min_{x \in X_1} \alpha = \max \left(\max_{k \in K^{\geq}} \left(\frac{\bar{f}_k - f_k(x^*)}{\left|\bar{f}_k - f_k\right|} \right), \max_{k \in K^{\leq}} \left(\frac{\tilde{f}_k - f_k(x^*)}{\left|\tilde{f}_k - f_k\right|} \right), \max_{k \in K^{>}} \left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} \right), \max_{k \in K^{<}} \left(\frac{f_k^* - f_k(x^*)}{\left|f_k^* - f_k\right|} \right) \right) .$$

because in the opposite case it could be decreased further. The right side of (34) can be written as:

$$\min_{x\in X_1} \alpha = \min_{x\in X_1} \left(\max \left(\max_{k\in K^{\geq}} \left(\frac{\bar{f}_k - f_k\left(x^*\right)}{\left|\bar{f}_k - f_k\right|} \right), \max_{k\in K^{\leq}} \left(\frac{\tilde{f}_k - f_k\left(x^*\right)}{\left|\tilde{f}_k - f_k\right|} \right), \right.\right.$$

$$\left.\left. \max_{k\in K^{>}} \left(\frac{f_k^* - f_k\left(x^*\right)}{\left|f_k^* - f_k\right|} \right), \max_{k\in K^{<}} \left(\frac{f_k^* - f_k\left(x^*\right)}{\left|f_k^* - f_k\right|} \right) \right) \right),$$

which proves the theorem.

Every equivalent problem of the scalarizing problem RNS2, denoted by RNS2e, can be presented in this way:

(35) $$\min \left(\alpha + \delta \sum_{k\in K} y_k \right)$$

under the constraints:

(36) $$\alpha \geq \frac{\overline{f}_k - f_k(x)}{\left|\overline{f}_k - f_k\right|}, \ k \in K^{\geq},$$

(37) $$\alpha \geq \frac{\tilde{f}_k - f_k(x)}{\left|\tilde{f}_k - f_k\right|}, \ k \in K^{\leq},$$

(38) $$\alpha \geq \frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|}, \ k \in K^{<},$$

(39) $$\alpha \geq \frac{f_k^* - f_k(x)}{\left|f_k^* - f_k\right|}, \ k \in K^{>},$$

(40) $$\bar{f}_k - f_k(x) = y_k, \ k \in K^{\geq},$$

(41) $$\tilde{f}_k - f_k(x) = y_k, \ k \in K^{\leq},$$

(42) $$f_k - f_k(x) = y_k, \quad k \in K^{<} \cup K^{>},$$

(43) $$- f_k(x) \geq f_k, \ k \in K^{>},$$

(44) $$f_k(x) \geq \tilde{f}_k, \ k \in K^{\leq},$$

(45) $$x \in X_1,$$

(46) $$\alpha, y_k \ k \in K \text{ – arbitrary.}$$

The scalarizing problem RNS2e has the same properties as problem RNS1a, but it has more constraints and variables, because this problem is more difficult for solving.

4. Concluding remarks

The scalarizing problems RNS are formulated based on implicit classification of the criteria, defined by the DM. With this classification, the DM sets his/her desired alterations of the criteria values in reference with the current preferred solution. These scalarizing problems can be examined as a modification of the proposed classification-oriented scalarizing problems (N a r u l a, V a s s i l e v [7], V a s s i l e v a [11], V a s s i l e v et al. [10], V a s s i l e v a [12], V a s s i l e v a et al.[13]), which also utilize implicit classification (partition) of the criteria in groups. The scalarizing problems RNS possess most of the positive properties of these problems. The greater freedom, which is given to the DM to express his/her local preferences, enables the DM to be more efficient in finding the most preferred solution and to feel more confident about the quality of this solution. In the general case (from the mathematical point of view) the current preferred solution and the local preferences of the DM define a comparatively narrow reference-neighborhood in the non-dominated set. The feasible solutions of the integer scalarizing problem lie comparatively close to the efficient (Pareto optimal) surface of the MLIP. This enables the use of heuristic integer algorithms (G l o v e r, L a g u n a [3]) for its solution. On the other hand, the DM works mainly in the criteria space when applying these scalarising problems. Since the criteria of most of the problems have physical or financial interpretation, this feature allows him/her to judge, choose and take the most realistic decision.

R e f e r e n c e s

1. B e n a y o u n, R., J. de M o n t g o l f i e r, J. T e r g n y, O. L a r i c h e v. Linear Programming with Multiple Objective Functions: Step Method (STEM). – Mathematical Programming, **1**, 1971, 366-375.
2. B u c h a n a n, J. T. A Naive Approach for Solving MCDM Problems: The GUESS Method. – Journal of the Operational Research Society, **48**, 1997, 202-206.
3. G l o v e r, F., M. L a g u n a. Tabu Search. Boston, Dordrecht, London, Kluver Academic Publisher, 1997.
4. K o r h o n e n, P. Reference Direction Approach to Multiple Objective Linear Programming : Historical Overview, Essay in Decision Making: A Volume in Honour of Zionts (M.H. Kanwan, J. Spronk, J. Walleniius, Eds.). Berlin, Springer-Verlag, 1997, 74-92.
5. M i e t t i n e n, K. Nonlinear Multiobjective Optimization. Boston, Kluwer Academic Publishers, 1999.
6. N a k a y a m a, H., Y. S a w a r a g i. Satisficing Trade-Off Method for Multiobjective Programming. – In: Interactive Decision Analysis (M. Grauer, A. Wierzbicki, Eds.). – Lecture Notes in Economics and Mathematical Systems, **229**, Springer-Verlag, 1984, 113-122.
7. N a r u l a, S. C., V. V a s s i l e v. An Interactive Algorithm for Solving Multiple Objective Integer Linear Programming Problems. – European Journal of Operational Research, **79**, 1994, 443-450.
8. N e m h a u s e r, G. L., L. A.W o l s e y. Integer and Combinatorial Optimization. New York, Wiley, 1988.
9. S t e u e r, R. E. The Tchebycheff Procedure of Interactive Multiple Objective Programming. – In: Multiple Criteria Decision Making and Risk Analysis, Using Microcomputers (B. Karpak, S. Zionts, Eds.). Berlin, Springer-Verlag, 1989, 235-249.

10. V a s s i l e v, V., S. N a r u l a, M. V a s s i l e v a, K. G e n o v a. A Partition-Based Interactive Algorithm for Solving Multicriteria Linear Integer Programming Problems. – Cybernetics and Information Technologies, **4**, 2004, No 2, 41-60.
11. V a s s i l e va , M. A Learning-Oriented Method of Linear Mixed Integer Multicriteria Optimization. – Cybernetic and Information Technologies, **4**, 2004, No 1, 13-25.
12. V a s s i l e v a, M. A Generalized Scalarizing Problem of Multicriteria Optimization. – Comp. Rend. de l'Acad. Bulg. des Sci., **58**, 2005, No 5, 537-544.
13. V a s s i l e v a, M., K. M i e t t i n e n, V. V a s s i l e v. Generalized Scalarizing Problem GENWS. IIT Working Papers IIT/WP-205, BAS, 2005.
14. W i e r z b i c k i, A. P. The Use of Reference Objectives in Multiobjective Optimization. – In: Multiple Criteria Decision Making: Theory and Application. Lecture Notes in Economics and Mathematical Systems (G. Fandel, T. Gal, Eds.). No 177. Springer-Verlag, 1980, 468-486.
15. W o l s e y, L. Integer Programming. Wiley-Interscience, 1988.