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Some Basic Invariant Properties of the Multidimensional Hough Transform[1]

Valeri Ilchev, Zlatoliliya Ilcheva***

**Institute of Information Technologies, 1113 Sofia,*
E-mail: ilch@iinf.bas.bg
***Institute of Computer and Communication Systems, 1113 Sofia,*
E-mail: zlat@iccs.isdip.bas.bg

Abstract: *The paper discusses the presentation in the multidimensional Hough space of a contour set and also the presentation of the two sets – external and internal, to which this contour divides the convex hypersurface, which is determined in the object space. Each point of these sets is represented by an indirect transform in the Hough space, i.e. by means of the supporting hyperplane (maybe not unique), related to the convex hypersurface in this point.*

The results of the theoretical research indicate, that in this representation, the basic properties of the sets are preserved. In the given case this means that in the object space the connected sets, which are the contour and the generated by it external and internal sets, correspond to the connected sets in the Hough space. These sets are parts of the convex hypersurface with the same mutual disposition, i.e. the contour set in the object space corresponds to a contour set in the Hough space. Analogously, the external and internal sets in the first space correspond to an external set and an internal one in the second space.

Keywords: *Multidimensional Hough transform, contour sets, convex sets, theory of sets, topological spaces.*

1. Introduction

If we define in the *n*-dimensional Euclidian space $\boldsymbol{E}^n$ two compact and not mutually crossing sets, then in the most general form their shape can be presented by their convex envelopes. It is clear that if the sets are linearly inseparable, then these

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envelopes will be crossed. Presented in this way the initial sets can be considered as two compact, convex and mutually crossing sets, and at all of their border points there exists at least one supporting hyperplane, related to the corresponding set.

In the case considered, the interesting item is the set $\boldsymbol{Z}$ obtained by the boundary's section of these two convex sets. The hyperplanes which are supporting, related to the corresponding convex set in the points of this section –$\boldsymbol{Z}$, have characteristic properties**,** which can be used for the initial construct of the optimally classificating hyperplane. Although the classificating hyperplane will cross the two given sets (if they are compact, but not convex) it can be considered as a first approximation of the optimal separating hypersurface for these sets. Each such hyperplane, defined in the space $\boldsymbol{E}^n$, may be uniquely presented in the Hough space: $\mathbf{L}^n$ where $n << \infty$. This presentation will be an uniquely reversible transform, where each hyperplane in $\boldsymbol{E}^n$, corresponds to a certain point in space $\mathbf{L}^n$, like the straight lines in the 2-dimensional Hough transform or the planes in the 3-dimensional similar transform [1, 2]. The images of the supporting hyperplanes in the space $\mathbf{L}^n$, to the corresponding convex sets, will define a set of points $\mathbf{H}_Z$, which will have also characteristic properties that are discussed in this paper.

In the general case, the investigation aims at ascertaining a correspondence between the basic properties of the set $\boldsymbol{Z}$ in $\boldsymbol{E}^n$ and the properties of its "indirect" transform – $\mathbf{H}_Z$, in the space $\mathbf{L}^n$. In this case under the notion "indirect" transform, we shall imply the representation of the points from $\boldsymbol{Z} \subset \boldsymbol{E}^n$ by this mapping in the space $\mathbf{L}^n$ of the supporting hyperplanes in these points to the corresponding convex set. In every point of the boundary of the convex set in $\boldsymbol{E}^n$ there exists at least one supporting hyperplane. Then it is obvious that if the supporting hyperplane is unique then at its boundary point from the convex set in $\boldsymbol{E}^n$ will correspond to an unique point in $\mathbf{L}^n$. Respectively if the supporting hyperplanes form a set, then the correspondence to their common (unique) supporting point in the convex set in E^n, will be a set of points in the space $\mathbf{L}^n$. It is clear from these facts, that unlike the multidimensional Hough transform which is homeomorphic [3], the representation in such way of the boundary points from some convex set $S \subset \boldsymbol{E}^n$ in the space $\mathbf{L}^n$, will not always be unique, i.e. it will not be a homeomorphism. For example as it is established in [4], if the border of two mutually crossing convex and compact sets defined in $\boldsymbol{E}^n$ are indirectly mapped (by their supporting hyperplanes) in the space $\mathbf{L}^n$, then their images will be the borders of two not mutually crossing convex, closed and infinite sets. This means that the intersection $\boldsymbol{Z}$ of the borders of both sets in $\boldsymbol{E}^n$, i.e. their common and unique set will be (indirectly) represented by two sets in the space $\mathbf{L}^n$; the one set will belong to the first boundary surface and the other set – to the second one.

In spite of these disparity in the case of an indirect mapping, as it is evident from Theorem 1 [4], some properties of the border of the convex set in $\boldsymbol{E}^n$ will be preserved in its representation in the space $\mathbf{L}^n$. According to this theorem, if we examine the border of S as a convex hypersurface P_S then we shall establish that this hypersurface in the space $\mathbf{L}^n$ will correspondingly be a border of a closed and convex (unlimited) set, i.e. this border will be convex (though unlimited) hypersurface $\mathbf{H}_S$ and this will mean that in this representation the "connectivity" property will be preserved. This fact allows us to set up several questions. For example if we define the contour set – $\boldsymbol{Z}$ over the hypersurface P_S and represent this set in $\mathbf{L}^n$ by the supporting (in the contour points) hyperplanes, then will this representation also be a contour over the hypersutface $\mathsf{H}_S \subset \mathsf{L}^n$? Another interesting point is: will the correspondence be preserved between the internal and the external subsets of P_S toward the contour $\boldsymbol{Z}$ for their representation in L^n, where these two subsets correspond to two parts of the hypersurface H_S.

The answer to these and other questions concerning the properties of this representation is the kernel of the theoretical research in the present paper. For greater clarity of the exposition and simplifying the theoretical research for the analysis of the cases of two mutually crossing convex and compact sets we shall assume about the type of the second convex set that it is a cone in $\boldsymbol{E}^n$, which is a convex and supporting to the first (convex) set, without violating the generality of the theoretical results.

2. Defining and investigation of the problem in the object space $\boldsymbol{E}^n$

Let us define in the n-dimensional Euclidian space $\boldsymbol{E}^n$ the convex, compact set S and point $\mathbf{x}_0$, such that $\mathbf{x}_0 \notin S$ and lying on one of the axis – $\boldsymbol{Y}$ of the space $\boldsymbol{E}^n$. If we assume that this point is an axis of a bunch of hyperplanes H_0, then we may define the set

$$\mathsf{H}_0 = \{H_0 \subset \boldsymbol{E}^n : H_0 \ni \mathbf{x}_0 ; H_0 \cap \mathrm{Int}(S) = \varnothing\}.$$

The boundary H_z of the set H_0, which is denoted with $\mathrm{Fr}(\mathsf{H}_0)$, may be specified in the following way:

$$\mathsf{H}_z = \mathrm{Fr}(\mathsf{H}_0) = \{H_z : \mathsf{H}_z \cap \mathrm{Fr}(S) \neq \varnothing;\ H_z \subset \mathsf{H}_0\}$$

and obviously will define the boundary of a supporting cone to S with an apex at the point $\mathbf{x}_0$, formed by the sections of $[H_z]^k$, where $[H_z]^k$ are the corresponding half-spaces of the supporting hyperplanes H_z to S, for which $S \subset [H_z]^k$, $k = \{\pm\}$, as it is shown in Fig. 1.

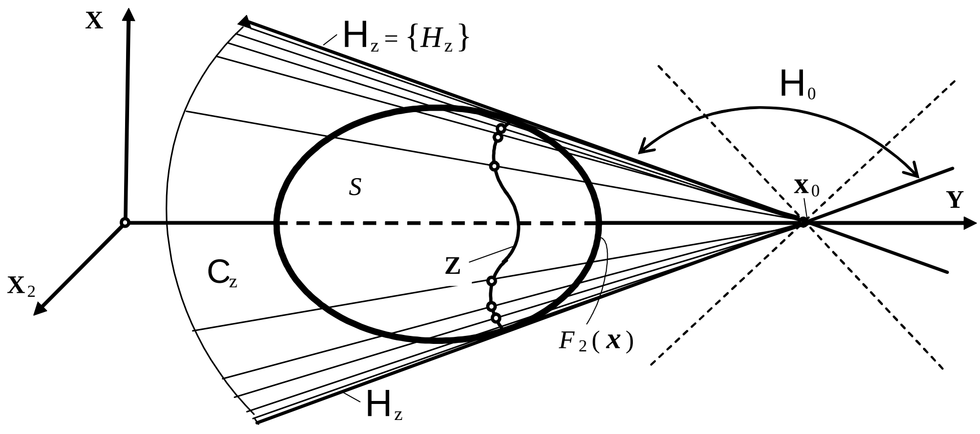

Fig.1. Set S and supporting to S cone C_z

Let us assign the set $\boldsymbol{Z}$, consisting of all points at which the hyperplanes H_z are supporting to S: $\boldsymbol{Z} = \{z : z \in H_z \cap \mathrm{Fr}(S);\ H_z \subset \mathsf{H}_z\}$. As by condition the set S is limited (because is compact) then $\boldsymbol{Z}$ will be a limited set too. Besides, further we ascertain the fact, that $\boldsymbol{Z}$ is a connected set at each one of its points.

Let us in the Euclidian space $\boldsymbol{E}^n$ separate the $(n-1)$-dimensional subspace $\mathbf{X}^{n-1} \subset \boldsymbol{E}^n$ and the axis $\boldsymbol{Y} \subset \boldsymbol{E}^n$. Then the boundary of the convex set S can be considered as a section of two functions $\boldsymbol{F}_1: \boldsymbol{X}^{n-1} \rightarrow \boldsymbol{Y}$ and $\boldsymbol{F}_2: \boldsymbol{X}^{n-1} \rightarrow \boldsymbol{Y}$, determinated by $n-1$ arguments belonging to $\boldsymbol{X}^{n-1}$: $\boldsymbol{F}_1(x_1, x_2, \ldots, x_{n-1})$ and $\boldsymbol{F}_2(x_1, x_2, \ldots, x_{n-1})$, where one of them is convex and the other concave in relation to the axis $\boldsymbol{Y}$. These two functions will specify the hypersurfaces P_{S1} and P_{S2} representing the two parts of boundary of the set S: Fr(S). Let us specify the close and a convex region $\boldsymbol{X}_S \subset \boldsymbol{X}^{n-1}$ so that for $\mathbf{x} \in \boldsymbol{X}_S$ the condition will be satisfied: $F_2(\boldsymbol{x}) = F_2(\boldsymbol{x})$, where: $F_2(\boldsymbol{x})$ is a concave function and $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1})$. Let besides $\boldsymbol{X}_Z \subset \boldsymbol{X}_S$, where $\boldsymbol{X}_Z = \mathrm{Pr}_X(\boldsymbol{Z})$ is the projection of the set $\boldsymbol{Z}$ on the subspace $\boldsymbol{X}^{n-1}$. The set $\boldsymbol{Z}$ will have the following property:

Property 2.1. The set $\boldsymbol{Z}$ is connected.

P r o o f. Since by condition the set $\boldsymbol{Z}$ is specified in the following way:

$\boldsymbol{Z} = \{z : z \in H_z \cap \mathrm{Fr}(S);\ H_z \cap \mathrm{Int}(S) = \varnothing\}$, then $\boldsymbol{Z} \subset C_z = \bigcap_i [H_z^i]$, where C_z is the supporting cone to S with an apex at the point $\mathbf{x}_0 \notin S$, and $[\dot{H}_z^i]$ – the half spaces of the hyperplanes $H_z^i \subset H_z$, [5]. Obviously the set C_z will be unlimited, close and convex (since $[H_z^i]$ are convex and close sets: $H_z^i \subset [H_z^i]$). In the region $\boldsymbol{X}_Z \subset \boldsymbol{X}_S$, the boundary of S – Fr(S) can be defined analytically, by the concave function $F_2(\boldsymbol{x})$, which is obviously continuous for $\boldsymbol{x} \in \boldsymbol{X}_Z$, i. e. $\mathrm{Fr}(S) = \{\mathbf{x} = (\boldsymbol{x}, y): \boldsymbol{x} \in \boldsymbol{X}_Z, \boldsymbol{y} = F_2(\boldsymbol{x})\}$ will be a connected set in the region $\boldsymbol{X}_Z$, where $\boldsymbol{X}_Z$ is a connected subset of the metric subspace $\mathbf{X}^{n-1}$. Since Fr(C_z) can be represented in the following way: $\mathrm{Fr}(C_z) = \bigcap_i [H_z^i]$, $i = 1, 2, \ldots$, and by condition $\forall z_i \in \mathrm{Fr}(C_z)$, then $\boldsymbol{Z} \subset \mathrm{Fr}(C_z)$. But by condition, we also have: $\forall z_i \in \mathrm{Fr}(S) \Rightarrow \boldsymbol{Z} \subset \mathrm{Fr}(C_z) \cap \mathrm{Fr}(S) \neq \varnothing$. It is clear that since $C_z = \overline{C_z}$ and $S = \overline{S}$ are connected sets, then $\boldsymbol{Z}$ will also be a connected set (since: $\boldsymbol{Z} \subset \overline{C_z}$, $\boldsymbol{Z} \subset \overline{S}$) and in this way Property 2.1 is completely proven ■

Let us define by means of the set $\boldsymbol{Z}$, the set representing the hypersurface $P_z \subset P_{S2}$, specified in the region $\boldsymbol{X}_S \subset \boldsymbol{X}^{n-1}$ in the following way: $P_z = \{\mathbf{x}_z \in \{\boldsymbol{x}_\lambda, F(\boldsymbol{x}_\lambda)\}: F(\boldsymbol{x}_\lambda) \geq \lambda F(\boldsymbol{x}_a) + (1-\lambda) F(\boldsymbol{x}_b);\ \boldsymbol{x}_a, \boldsymbol{x}_b \in \boldsymbol{X}_Z \subset \boldsymbol{X}_S\}$, where: $[\boldsymbol{x}_a, F(\boldsymbol{x}_a)] = z_a \in \boldsymbol{Z}$, $[\boldsymbol{x}_b, F(\boldsymbol{x}_b)] = z_b \in \boldsymbol{Z}$; $\boldsymbol{x}_\lambda = \lambda \boldsymbol{x}_a + (1-\lambda)\boldsymbol{x}_b$, $\lambda = [0, 1]$. If the set P_z is denoted by P, then for the projection of this set on the subspace $\boldsymbol{X}^{n-1}$, the following property will hold:

Property 2.2. The projection $\mathrm{Pr}_X(P)$ of P on the subspace $\boldsymbol{X}^{n-1}$ is a convex set.

P r o o f. Let us consider again the cone $C_z = \bigcap [H_z^i]$. Since $[H_z^i]$ are convex sets then C_z is a convex set too. It is clear, that $P \subset \dot{C}_z$, since $P \subset \overline{S}$ and $\overline{S} = S \subset C_z$. Along with this hyp$F_2(\boldsymbol{x})$ is a convex set, because by condition $F_2(\boldsymbol{x})$ for $\boldsymbol{x} \subset \boldsymbol{X}_S$ is a concave function. Let us take two points $\boldsymbol{x}_a, \boldsymbol{x}_b \in \boldsymbol{X}_Z \subset \boldsymbol{X}_S$. Then for $\boldsymbol{x}_\lambda = \lambda \boldsymbol{x}_a + (1-\lambda)\boldsymbol{x}_b$, $\lambda = [0, 1]$ we will have: $F(\boldsymbol{x}) \geq \lambda F(\boldsymbol{x}_a) + (1-\lambda) F(\boldsymbol{x}_b) \Rightarrow \mathbf{x}_\lambda = [\boldsymbol{x}_\lambda, F(\boldsymbol{x}_\lambda)] \in P \subset C_z$. Since these conditions are fulfilled for every projection $\boldsymbol{x}_\lambda$ of the point $\mathbf{x}_\lambda$, where $\{\mathbf{x}_\lambda\} \subset \mathrm{hyp}F(\boldsymbol{x})$, then the set $\{\boldsymbol{x}_\lambda\} = \mathrm{Pr}_X(P)$ is obviously convex and Property 2.2 is completely proven ■

If we juxtapose the sets P and $\boldsymbol{Z}$, then $\boldsymbol{Z}$ may be viewed as a contour set of P, according to the following definition:

Definition 2.1. Let a set $\boldsymbol{G}$ be given as well as its subsets: $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C} \subset \boldsymbol{G}$. We will call $\boldsymbol{C}$ a contour set of $\boldsymbol{A}$ and $\boldsymbol{B}$ (for shortness – contour of $\boldsymbol{A}$ and $\boldsymbol{B}$), if these sets fulfill the following conditions: 1) $\mathrm{Fr}(\boldsymbol{A}) = \boldsymbol{C} = \mathrm{Fr}(\boldsymbol{B})$; 2) The set $\boldsymbol{G} \setminus \boldsymbol{C}$ consists only of two sets: Int($\boldsymbol{A}$) and Int($\boldsymbol{B}$), i. e. $\overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}} = \boldsymbol{G}$.

From this definition the following property of the contour $\boldsymbol{C}$ can be derived [3]: each arc $\overset{\frown}{ab}$, defined by the points $a \in \mathrm{Int}(\boldsymbol{A})$ and $b \in \mathrm{Int}(\boldsymbol{B})$ will be such that $\overset{\frown}{ab} \cap \mathbf{C} \neq \varnothing$, or which is the same: $\overline{\exists}\,(\overset{\frown}{ab} \cap \boldsymbol{C}) = \varnothing$, where an arc is each set, which is homeomorphous to the close interval [0, 1]. Then for $\boldsymbol{Z}$, we can prove the following property:

Property 2.3. The set $\boldsymbol{Z}$ is a common contour of the sets $\overline{P}$ and $\overline{P_s}$, where: $\overline{P}$, $\overline{P_s} \subset P_{S2}$ and $P_s = P_{S2} \setminus P$; $P = \overline{P}$, $P_s = \mathrm{Int}(\overline{P_s})$.

P r o o f. Let us consider the hypersurface $P \subset P_{S2}$ in Fig. 2 and denote its projection on the subspace $\boldsymbol{X}^{n-1}$ in the following way: $\mathrm{Pr}_X(P) = \boldsymbol{X}_p$. According to Property 2. 2, this projection is a convex set, for which we have: $\boldsymbol{X}_p \subset \boldsymbol{X}_S$.

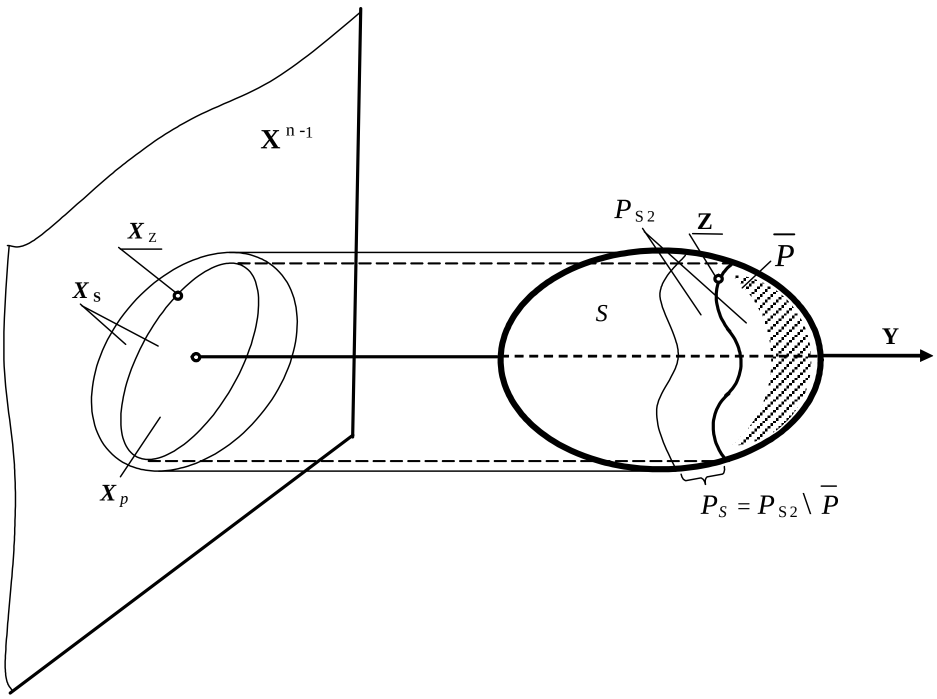

Fig. 2. The hypersurfaces P, P_S and P_{S2} and their projections on the subspace $\boldsymbol{X}^{n-1}$

Since $\boldsymbol{X}_S$ may be chosen so that, $\boldsymbol{X}_p \subset \mathrm{Int}(\boldsymbol{X}_S)$, where $\boldsymbol{X}_p$ is a close set because the set $\boldsymbol{Z}$ participates in the definition of P, from where: $\mathrm{Fr}(\boldsymbol{X}_p) = \boldsymbol{X}_Z = \mathrm{Pr}_X(\boldsymbol{Z})$: Fig. 2, then $\boldsymbol{X}_Z \subset \mathrm{Int}(\boldsymbol{X}_S)$. The set $\boldsymbol{X}_Z$ will be a connected set, because it is a boundary of the convex set $\boldsymbol{X}_p$. Then, for each of its points $\boldsymbol{x}_Z \subset \boldsymbol{X}_Z$ there will be a surrounding area $\mathrm{O}(\boldsymbol{x}_Z) \subset \boldsymbol{X}_S$ such that $\mathrm{O}(\boldsymbol{x}_Z) \cap \mathrm{Int}(\boldsymbol{X}_p) \neq \varnothing$ and $O(\boldsymbol{x}_Z) \cap \mathrm{Int}(\boldsymbol{X}_s) \neq \varnothing$, for $\boldsymbol{X}_s = \boldsymbol{X}_S \setminus \boldsymbol{X}_p$. If in this surrounding area we take two points $\boldsymbol{x}_a$, $\boldsymbol{x}_b \in \mathrm{O}(\boldsymbol{x}_Z)$, $\boldsymbol{x}_a$, $\boldsymbol{x}_b \notin \boldsymbol{X}_Z$, where $\boldsymbol{x}_a \in O(\boldsymbol{x}_Z) \cap \mathrm{Int}(\boldsymbol{X}_p)$, and $\boldsymbol{x}_b \in O(\boldsymbol{x}_Z) \cap \mathrm{Int}(\boldsymbol{X}_s)$, then the arc $\overset{\frown}{\boldsymbol{x}_a \boldsymbol{x}_b}$ ($\boldsymbol{x}_Z \in \overset{\frown}{\boldsymbol{x}_a \boldsymbol{x}_b}$), determined by these points, will obviously cross $\boldsymbol{X}_Z$ in the surrounding area $\mathrm{O}(\boldsymbol{x}_Z)$: $\overset{\frown}{\boldsymbol{x}_a \boldsymbol{x}_b} \cap \boldsymbol{X}_Z = \boldsymbol{x}_Z$. Since for each point $\boldsymbol{x}_Z \in \boldsymbol{X}_Z$ we can construct such a surrounding area with the same properties, then according to previous definition, the set $\boldsymbol{X}_Z$ will be a contour of $\boldsymbol{X}_p$ in the space $\boldsymbol{X}^{n-1}$. Obviously, for the concave hypersurface $P_{S2} = [\boldsymbol{x}_S, F(\boldsymbol{x}_S)]$, $\boldsymbol{x}_S \in \boldsymbol{X}_S$, a uniquely reversible mapping $\boldsymbol{F}$ will exist: $P_{S2} \rightarrow \boldsymbol{X}_S$ such that each point of the hypersurface P_{S2} will correspond to a certain point of the set $\boldsymbol{X}_S$ and vice versa. Let us specify the set $O_s(\boldsymbol{x}_Z)$: $O_s(\boldsymbol{x}_Z) = O(\boldsymbol{x}_Z) \cap \boldsymbol{X}_S = O(\boldsymbol{x}_Z) \cap (\boldsymbol{X}_S \backslash \boldsymbol{X}_p) = [O(\boldsymbol{x}_Z) \cap \boldsymbol{X}_S] \setminus \boldsymbol{X}_p = O(\boldsymbol{x}_Z) \setminus \boldsymbol{X}_p$; $(O(\boldsymbol{x}_Z) \subset \boldsymbol{X}_S)$ and the image $\boldsymbol{F}[O_s(\boldsymbol{x}_Z)]$:

$$\boldsymbol{F}[O_s(\boldsymbol{x}_Z)] = \boldsymbol{F}[O(\boldsymbol{x}_Z) \backslash \boldsymbol{X}_p] = \boldsymbol{F}[O(\boldsymbol{x}_Z)] \setminus \boldsymbol{F}(\boldsymbol{X}_p) = \boldsymbol{F}[O(\boldsymbol{x}_Z)] \setminus P \neq \varnothing.$$

Since $O_s(\boldsymbol{x}_Z) \subset \boldsymbol{X}_s$, then $\boldsymbol{F}[O_s(\boldsymbol{x}_Z)] \subset \boldsymbol{F}(\boldsymbol{X}_s) = P_s$, where $P_s = \boldsymbol{F}(\boldsymbol{X}_S) \backslash \boldsymbol{F}(\boldsymbol{X}_p) = P_{S2} \backslash P$. If we come back to the points $\boldsymbol{x}_a$ and $\boldsymbol{x}_b$, then $\boldsymbol{x}_a \in \boldsymbol{X}_p \Rightarrow \boldsymbol{F}(\boldsymbol{x}_a) = \mathbf{x}_a \in \boldsymbol{F}(\boldsymbol{X}_p) = P$ and $\boldsymbol{x}_b \in \boldsymbol{X}_s \Rightarrow \boldsymbol{F}(\boldsymbol{x}_b) = \mathbf{x}_b \in \boldsymbol{F}(\boldsymbol{X}_s) = P_s$. Since the points $\boldsymbol{x}_a$ and $\boldsymbol{x}_b$ are chosen in such a way that $\boldsymbol{x}_a, \boldsymbol{x}_b \notin \boldsymbol{X}_Z$, then $\mathbf{x}_a, \mathbf{x}_b \notin \boldsymbol{Z}$ and from the condition $P_S = P_{S2} \setminus P$ it follows that $\mathbf{x}_a \notin P_s$, $\mathbf{x}_b \notin P$. Let us define in the space $\boldsymbol{E}^n$ the arc $\breve{\mathbf{x}_a \mathbf{x}_b} = \boldsymbol{F}(\breve{\boldsymbol{x}_a \boldsymbol{x}_b})$. From the condition $\boldsymbol{x}_Z \in \breve{\boldsymbol{x}_a \boldsymbol{x}_b}$ we will have $\boldsymbol{F}(\boldsymbol{x}_Z) = \boldsymbol{z} \in \breve{\mathbf{x}_a \mathbf{x}_b}$, i. e. this arc will contain a point of the set $\boldsymbol{Z} \Rightarrow \breve{\mathbf{x}_a \mathbf{x}_b} \cap \boldsymbol{Z} \neq \varnothing$, where $\mathbf{x}_a \in \mathrm{Int}(P)$ (because $\mathbf{x}_a \in P \wedge \mathbf{x}_a \notin \boldsymbol{Z} = \mathrm{Fr}(P)$, $P = \overline{P}$) and $\mathbf{x}_b \in \mathrm{Int}(P_s)$ (because $P_s = P_{S2} \setminus P$ is an open set). This means, that the arc $\breve{\mathbf{x}_a \mathbf{x}_b}$ will fulfill the condition of Definition 2.1. and taking into account that for each point of the set $\boldsymbol{Z}$ such an arc can be defined, then according to this definition, $\boldsymbol{Z}$ will be a common contour of the sets $\overline{P}$ and $\overline{P}_S$. In this way Property 2.3. is completely proven ■

By means of the contour $\boldsymbol{Z}$, we can represent the two sets P and P_s as covers compounded by arcs. Let us take two points $\boldsymbol{z}_a$ and $\boldsymbol{z}_b \in \boldsymbol{Z}$ and give the line $L_{ab} \subset P$ starting at point $\boldsymbol{z}_a$ and ending at point $\boldsymbol{z}_b$: $L_{ab}(\lambda) = [\boldsymbol{x}_\lambda, F(\boldsymbol{x}_\lambda)]$, where $\boldsymbol{x}_\lambda = \lambda \boldsymbol{x}_a + (1-\lambda)\boldsymbol{x}_b$, $\lambda = [0, 1]$, $\boldsymbol{x}_a = \mathrm{Pr}_X(\boldsymbol{z}_a)$, $\boldsymbol{x}_b = \mathrm{Pr}_X(\boldsymbol{z}_b)$. Obviously, each such line $L_{ab}(\lambda)$, will be homeomorphically represented on the segment line $\overline{\boldsymbol{z}_a \boldsymbol{z}_b} = (\boldsymbol{x}_\lambda, F_\lambda)$, where $F_\lambda = \lambda F(\boldsymbol{x}_a) + (1-\lambda) F(\boldsymbol{x}_b)$, and so on the close interval $0 \le \lambda \le 1$, i. e. $L_{ab}(\lambda)$ will be an arc in P. For the connected set $\boldsymbol{Z}$ we will have an unlimited number of pairs of points $\boldsymbol{z}^i_a$ and $\boldsymbol{z}^i_b$, whose arcs $L_i \subset P$ will form a cover of the set P [6; 7, Vol. II.]: $P = \bigcup_{i=1}^{\infty} L_i = \boldsymbol{L}_\mathrm{p}$. Since each arc begins at some point $\boldsymbol{z}^i_a$ and ends at some point $\boldsymbol{z}^i_b$, where $\boldsymbol{z}^i_a$ and $\boldsymbol{z}^i_b \in \boldsymbol{Z}$ ($\boldsymbol{Z} = \mathrm{Fr}(P)$), then $\boldsymbol{L}_p$ and P will also be compact sets, whence we will call P an internal set in relation to the contour $\boldsymbol{Z}$.

By analogy we can determine the arcs $L_{cd} \subset P_s$ beginning at point $\boldsymbol{z}_c \in \boldsymbol{Z}$ and ending at point $\mathbf{x}_d$, where $\mathbf{x}_d = [\boldsymbol{x}_d, F(\boldsymbol{x}_d)]$, $\boldsymbol{x}_d \in \mathrm{Fr}(\boldsymbol{X}_S) \subset \boldsymbol{X}^{n-1}$, i. e. the $\mathbf{x}_d$ is an end point for the region $\boldsymbol{X}_S$, in which the hypersurface $P_s \subset P_{S2}$ is defined. Such pairs of points will also be unlimited in number and we can form the covering of P_s of their arcs $L_j = L_{cd} \subset P_s$, in the following way: $P_s = \bigcup_{i=1}^{\infty} L_i = \boldsymbol{L}_s$. In contrast to $\boldsymbol{L}_p$ the covering $\boldsymbol{L}_s$ may be continued [7, Vol. I.]. Setting the regions $\boldsymbol{X}_\mathrm{S}$ in the following way: $\boldsymbol{X}^1_S \subset \boldsymbol{X}^2_S \subset \ldots \subset \boldsymbol{X}^m_S = \boldsymbol{X}_S$ (in the space $\mathbf{X}^{n-1}$) we will obtain in the space $\boldsymbol{E}^n$ a sequence of embedded covers: $\boldsymbol{L}^1_S \subset \boldsymbol{L}^2_S \subset \ldots \subset \boldsymbol{L}^m_S = \boldsymbol{L}_s$, where $\boldsymbol{L}_s = P_s$ for the region $\boldsymbol{X}_S$, i. e. if the region $\boldsymbol{X}_S$ is not a compact set in $\boldsymbol{X}^{n-1}$ then $\boldsymbol{L}_s$ and P_s also will not be compact sets in $\boldsymbol{E}^n$ (unlike P). Because of this we will call the set P_s external set toward the contour $\boldsymbol{Z}$. For this reason, it is clear that the contour $\boldsymbol{Z}$ will divide the hypersurface P_{S2} (considered as a set) in to two subsets – P and P_s which will be two and only two components of the set $P_{S2}\backslash\boldsymbol{Z}$, where: $\mathrm{Fr}(P) = \boldsymbol{Z} = \mathrm{Fr}(P_s)$.

Let us define in the space $\boldsymbol{E}^n$ the set H_p, compound of the subset of the supporting to S hyperplanes in the following way:

$$\mathsf{H}_p = \{H_p \supset \{\mathbf{x}_p\}: \{\mathbf{x}_p\} \subset P\,;\ \mathbf{x}_p = H_p \cap \mathrm{Fr}(S) \neq \varnothing,\ H_p \cap \mathrm{Int}(S) = \varnothing\}.$$

Then for this set we can formulate the following properties, which will be further used in the research:

Property 2.4. Let for the hyperplane H_t in the space $\boldsymbol{E}^n$ the following conditions be satisfied: $H_t \cap \mathrm{Fr}(S) \neq \varnothing$ and $H_t \cap \mathrm{Int}(S) = \varnothing$. If the axis $\boldsymbol{Y}$ intersects H_t in the point $\mathbf{x}_t^y = (\mathbf{0}, y_t)$ $y_s \le y_t \le y_0$, where for $P \subset \mathrm{Fr}(S)$: $\mathbf{x}_s = (\mathbf{0}, y_s) = P \cap \boldsymbol{Y}$ and $\mathbf{x}_0 = (\mathbf{0}, y_0) \in \boldsymbol{Y}$, ($\mathbf{x}_0$ – apex of the cone C_z, $\mathbf{x}_0 \notin S$), then for the set of points $\boldsymbol{t} = H_t \cap \mathrm{Fr}(S)$ the inclusion $\boldsymbol{t} \subset P$ will be valid.

The following property will be opposite to Property 2.4:

Property 2.5. Let $H_t \subset \boldsymbol{E}^n$ be a supporting hyperplane to the compact set S so that with the exception of the points $\{\mathbf{x}_t\} = \boldsymbol{t} = H_t \cap \mathrm{Fr}(S)$ will be fulfilled the condition $S \subset [H_t^-]$, where $[H_t^-]$ is the negative half space of the hyperplane $\boldsymbol{H}_t$. If the set $\boldsymbol{t}$ (which can consist of only one point $\mathbf{x}_t$) is such that $\boldsymbol{t} \subset P$, then for the cross point $\mathbf{x}_t^y = (\mathbf{0}, y_t) = H_t \cap \boldsymbol{Y}$ we will have: $y_s \le y_t \le y_0$, where $\mathbf{x}_s = (\mathbf{0}, y_s)$, $y_s = P \cap \boldsymbol{Y}$; $P \subset \mathrm{Fr}(S)$ and $\mathbf{x}_0 = (\mathbf{0}, y_0) \in \mathbf{Y}$, $\mathbf{x}_0 \notin S$ ($\mathbf{x}_0$ – apex of the cone C_z).

Accounting Property 2. 4 we can formulate and prove the corollary:

Corollary 2.4.1. If for the compact set $S \subset \boldsymbol{E}^n$ two supporting cones C_a and C_b are given to it, which apexes $\mathbf{x}_{0a} = (\mathbf{0}, y_{0a})$ and $\mathbf{x}_{0b} = (\mathbf{0}, y_{0b})$, belong to the axis $\boldsymbol{Y}$ and such that $y_s \le y_{0a} \le y_{0b}$, where $\mathbf{x}_s = (\mathbf{0}, y_s)$, $y_s = P_a \cap \boldsymbol{Y}$; $P_a \subset \mathrm{Fr}(S)$, then for the set $\boldsymbol{Z}_a$ of the supporting points of the cone C_a to S and P_b – the internal set of the set $\boldsymbol{Z}_b$, where $\boldsymbol{Z}_b$ is the set analogous to $\boldsymbol{Z}_a$ (for the cone C_b), will be valid the condition: $\boldsymbol{Z}_a \subset P_b$.

P r o o f. Let the hyperplane H_a^i be such that $H_a^i \subset \mathsf{H}_{za}$, i. e. $\mathbf{x}_{oa} \in H_a^i$, where: $\mathsf{H}_{za} = \{H_a \ni \mathbf{x}_{oa}: H_a \cap \mathrm{Fr}(S) \neq \varnothing, H_a \cap \mathrm{Int}(S) = \varnothing\}$. Then for the set $z_a^i = H_a^i \cap \mathrm{Fr}(S)$ according to Property 2.4 we will have $z_a^i \subset P_b$. Since the set $\boldsymbol{Z}_a$ is a connected set:

$$\boldsymbol{Z}_a = \{ z_a^i : z_a^i \in H_a^i \cap \mathrm{Fr}(S) \neq \varnothing;\ H_a^i \cap \mathrm{Int}(S) = \varnothing\},$$

then it can be defined in the following way: $\boldsymbol{Z}_a = \bigcup_i z_a^i$, from where directly the condition follows: $\boldsymbol{Z}_a \subset P_b$ and Corollary 2.4.1 is completely proven ■

3. Investigation of the problem in the Hough space L^n

Let in conjunction with H_z and H_p, the sets H_{S2} and H_s, are considered, the compounds of all hyperplanes, which are supported to the hypersurfaces P_{S2} and P_s:

$$\mathsf{H}_{S2} = \{H_{S2}: H_{S2} \ni \mathbf{x}_{S2}, \mathbf{x}_{S2} \in P_{S2}; H_{S2} \cap \mathrm{Fr}(S) \neq \varnothing, H_{S2} \cap \mathrm{Int}(S) = \varnothing\};$$

$\mathbf{x}_{S2} \in \boldsymbol{X}_S$,

$\mathbf{x}_{S2} = [\boldsymbol{x}_{S2}, F(\boldsymbol{x}_{S2})]$, and $\mathsf{H}_s = \{H_s: H_s \ni \mathbf{x}_s, \mathbf{x}_s \in P_s; H_s \cap \mathrm{Fr}(S) \neq \varnothing, H_s \cap \mathrm{Int}(S) = \varnothing\}$.

For the images $T(\mathsf{H}_z)$, $T(\mathsf{H}_p)$, $T(\mathsf{H}_{S2})$ and $T(\mathsf{H}_s)$ of these sets in the space $\mathbf{L}^n$, the following theorem will be correct:

Theorem 3.1. Let in the space $\mathbf{L}^n$ the sets H_z, H_p and $\mathsf{H}_s \subset \mathsf{H}_{S2} \subset \boldsymbol{E}^n$ are mapping: $\mathbf{H}_z = T(\mathsf{H}_z)$, $\mathbf{H}_p = T(\mathsf{H}_p)$, $\mathbf{H}_{S2} = T(\mathsf{H}_{S2})$, $\mathbf{H}_s = T(\mathsf{H}_s)$. Then the set $\mathbf{H}_z$ will be a common contour of the sets $\mathbf{H}_p$ and $\mathbf{H}_s$ in the space $\mathbf{L}^n$, where:

$$\mathrm{Fr}(\mathbf{H}_p) = \mathbf{H}_z = \mathrm{Fr}(\mathbf{H}_s), \text{ and } \mathbf{H}_{S2} = \mathbf{H}_p \cup \mathbf{H}_z \cup \mathbf{H}_s.$$

P r o o f: See **Appendix A**.

From Theorem 3.1. we can formulate the corollary:

Corollary T.3.1. The contour $\mathbf{H}_z$ separates the sets $\mathbf{H}_s = T(\mathsf{H}_s)$ and $\mathbf{H}_p = T(\mathsf{H}_p)$ in the space $\mathbf{L}^n$.

P r o o f . The proof is obvious, bearing in mind the condition of Theorem 3.1. and Definition 2.1.

From this corollary and from a Theorem 3.1 it is clear, that since the contour $\boldsymbol{Z}$ which, in the space $\boldsymbol{E}^n$, separates the hypersurface P_{S2} in two parts – external P_s and internal P, then the contour $\mathbf{H}_z$ will separate (in the space $\mathbf{L}^n$) the hypersurface $\mathbf{H}_{S2}$ in two sets – external and internal too. These sets are $\mathbf{H}_s$ and $\mathbf{H}_p$ according to the previous notations in Theorem 3.1. and Corollary T.3.1., but the question: which of them will be an external and correspondently – internal set towards the contour $\mathbf{H}_z$, must be considered additionally.

This correspondence is determined from the following statement:

Statement 3.1. The set $\mathbf{H}_p = T(\mathsf{H}_p)$ will be internal and the set $\mathbf{H}_s = T(\mathsf{H}_s)$ – external, towards the contour $\mathbf{H}_z$ in the space $\mathbf{L}^n$.

P r o o f. See **Appendix B**.

4. Conclusion

The theoretical results obtained in the article will be valid for the more common cases too, where, instead of the set S, two compact, convex and mutually intersecting sets S_1 and S_2 may be considered. For example, the set $\boldsymbol{Z}$: $\boldsymbol{Z} = \mathrm{Fr}(S_1) \cap \mathrm{Fr}(S_2) \neq \varnothing$ will be a common contour for both hypersurfaces P^1_{S2} and P^2_{S2}, which are the boundaries of the sets S_1 and S_2. The contour $\boldsymbol{Z}$ will separate each of these hypersurfaces into two subsets – external ones: P^1_{S2}, P^2_{S2}, and internal ones: P_1, P_2. In this case the connectivity of the set $\boldsymbol{Z}$ is proved analogously to Property 2.1, since by condition S_1 and S_2 are closed and convex (and so connected) sets.

Theorem 3.1 as well as Statement 3.1 will be also valid for the cases of two crossing one another and convex sets S_1 and S_2. From this, we conclude that the obtained results preserve in this aspect their universality, in spite of the simpler theoretical presentation of the simplified task considered in this paper.

The results of this paper are applied to investigate the learning process of a classifying neural network (NN). The coefficients of the NN, which are subject to the optimization, form the Hough space. Some basic properties of this space are identical with the properties of the object space. Their usage can substantially facilitate the theoretical researches of the NN in many different aspects.

Appendix A

Proof of **Theorem 3.1.** Let us suppose that in the space $\mathbf{L}^n$, the set $\mathbf{H}_z$ is not a contour in $\mathbf{H}_{S2}$, i.e. the set $\mathbf{H}_z$ does not separate $\mathbf{H}_{S2}$ in two subsets $\mathbf{H}_p$ and $\mathbf{H}_s$. Then, since $\mathbf{H}_{S2}$ is compound of the connected sets $\mathbf{H}_p$, $\mathbf{H}_z$ and $\mathbf{H}_s$ (the homeomorphic transforms T of the connected sets are connected sets too), then $\mathbf{H}_p \cup \mathbf{H}_s$ will be a connected set, in which an arc $\overset{\frown}{h_a h_b}$ can be given such that, the points h_a and h_b are not separated from the set $\mathbf{H}_z$ i.e.: $\mathbf{H}_z \cap \overset{\frown}{h_a h_b} = \varnothing$. Let us give in the set $\mathbf{H}_{S2}$ two sections $\mathbf{H}_1 = \boldsymbol{H}_1 \cap \mathbf{H}_{S2} \neq \varnothing$ and $\mathbf{H}_2 = \boldsymbol{H}_2 \cap \mathbf{H}_{S2} \neq \varnothing$, of the set $\mathbf{H}_{S2}$ with the hyperplanes $\boldsymbol{H}_1 = \{h_1 : h_1 = (\boldsymbol{c}_1, L_1), L_1 = \text{const}\}$ and $\boldsymbol{H}_2 = \{h_2 : h_2 = (\boldsymbol{c}_2, L_2), L_2 = \text{const}\}$, i.e.$\boldsymbol{H}_1 || \boldsymbol{H}_2 || \boldsymbol{C}^{n-1}$, where $\boldsymbol{C}^{n-1}$ is $(n-1)$-dimensional subspace in $\mathbf{L}^n$. If we choose L_1 and L_2 such that $L_1 > \sup\{h_z(\boldsymbol{c}, L_z): h_z \in \mathbf{H}_z\}$ and $L_1 < \inf\{h_z(\boldsymbol{c}, L_z): h_z \in \mathbf{H}_z\}$, where $L_1 > L_2$ and the points h_a and h_b are such that: $h_a \in \text{Fr}(\mathbf{H}_1)$, $h_b \in \text{Fr}(\mathbf{H}_2)$, then for the reverse transformations of these points in the space $\boldsymbol{E}^n$ we will have: $T^{-1}(h_a) = H_a \subset T^{-1}[\text{Fr}(\mathbf{H}_1)] = \mathsf{H}_1$ and $T^{-1}(h_b) = H_b \subset T^{-1}[\text{Fr}(\mathbf{H}_2)] = \mathsf{H}_2$. In the given case, the sets H_1 and H_2 will be the borders of two cones $\mathbf{x}_{01} = (\mathbf{0}, y_1) \in \boldsymbol{Y}$ and $\mathbf{x}_{02} = (\mathbf{0}, y_2) \in \boldsymbol{Y}$, for which, by condition we will have $y_1 > y_2$, where $y_1 = L_1$ and $y_2 = L_2$. Besides, from the two conditions $L_1 > \sup\{h_z\}$ and $L_1 < \inf\{h_z\}$ it is clear that for each hyperplane H_z where $H_z \subset T^{-1}(\mathbf{H}_z) = \mathsf{H}_z$, the point of its intersection with the axis $\boldsymbol{Y}$: $\mathbf{x}_{z0} = (\mathbf{0}, y_z) = H_z \cap \boldsymbol{Y}$ will belong to the following set: $\{\mathbf{x}_z(\mathbf{0}, y_z): y_2 < y_z < y_1, y_z = L_z\}$. $\{\mathbf{x}_z(\mathbf{0}, y_z): y_2 < y_z < y_1, y_z = L_z\}$. Let us denote the sets, which are composite of the supporting points of the hyperplanes $\{H_1\} = \mathsf{H}_1$ and $\{H_2\} = \mathsf{H}_2$ to the set S, with $\{\mathbf{x}_a\} = \boldsymbol{Z}_a$ and $\{\mathbf{x}_b\} = \boldsymbol{Z}_b$.

Similar to the set $\boldsymbol{Z}$, the sets $\boldsymbol{Z}_a$ and $\boldsymbol{Z}_b$ will be contours, everyone of them separates the hypersurface P_{S2} in two parts – external and internal, with respect to the corresponding contour. Let us denote with P_a the nternal set, in relation to the contour $\boldsymbol{Z}_a$ and with P_b – the external set, in relation to contour $\boldsymbol{Z}_b$. Then according to Corollary 2.4.1 we will have $P \subset P_a$, from which: $\boldsymbol{Z} \subset P_a$ and $\boldsymbol{Z}_b \subset P$ (P is an internal set to the contour $\boldsymbol{Z}$). Since $\mathbf{H}_1$ may be chosen such that for H_1 the set $\boldsymbol{Z}_a$ will satisfy the condition: $\boldsymbol{Z}_a \cap \boldsymbol{Z} = \varnothing$, then $\boldsymbol{Z} \subset \text{Int}(P_a)$ and evidently $\overline{P_a} = P_a \cup \boldsymbol{Z}_a$ will be a closed neighborhood of the set $\boldsymbol{Z}$. Since $\boldsymbol{Z}$ is a contour and it is contained in $\overline{P_a}$, then $\boldsymbol{Z}$ will separate this set in two parts – internal P and external $\overline{P'_a} = \overline{P_a} \setminus \overline{P}$, from which: $\overline{P} \subset \text{Int}(P_a)$. Then from the conditions $\overline{P_a} \subset P_{S2}$ and $\overline{P} \subset P_{S2}$, follows the evident inclusion: $P_{S2} \setminus \text{Int}(P_a) \subset P_{S2} \setminus \overline{P}$, i. e. $\overline{P_a^s} \subset P_s$, where $\overline{P_a^s}$ = $P_{S2} \setminus \text{Int}(P_a)$ (in this case the formula: $C \setminus (A \cup B) = (C \setminus A) \setminus B \subset C \setminus A$ is used, where for $A \subset B$: $C \setminus (A \cup B) = C \setminus B \Rightarrow C \setminus B \subset C \setminus A$). From the inclusion $\boldsymbol{Z}_a \subset \overline{P_a^s}$, it follows: $\boldsymbol{Z}_a \subset P_s$. Since $\boldsymbol{Z}_b \subset P$, then the contour $\boldsymbol{Z}$ evidently will separate (besides that – strictly) the two sets $\boldsymbol{Z}_a$ and $\boldsymbol{Z}_b$. It is clear that as soon as $H_a \subset \mathsf{H}_1$ and $H_b \subset \mathsf{H}_2$, then for their supporting points $\{x_a\} = \boldsymbol{Z}_a$ and $\{x_b\} = \boldsymbol{Z}_b$ to the set S, we will have: $\{\mathbf{x}_a\} \subset P_s$ and $\{\mathbf{x}_b\} \subset P$.

Let us consider again the arc $\overset{\frown}{h_a h_b}$ in the space L^n. Since this arc by definition, is a connected set, then for each point $h_t \in \overset{\frown}{h_a h_b}$ we will have: $\lim_{i \to \infty} |h_i - h_t| = 0$, where $h_1, h_2, \dots, h_i, \dots \in \overset{\frown}{h_a h_b}$. Then in the space $\boldsymbol{E}^n$, in view of homeomorphic transform T, we will have for $i = 1, 2, \dots$, : $\lim_{i \to \infty} | T^{-1}(h_i) - T^{-1}(h_t)| = 0 \Rightarrow H_i \to H_t$, from where for the supporting points $\mathbf{x}_i$ and $\mathbf{x}_t$ of the hyperplanes H_i and H_t to the set S, we will obtain: $\boldsymbol{x}_i \to \boldsymbol{x}_t$ (which directly follows from the evident equation: $H_i \cap \text{Fr}(S) = H_t \cap \text{Fr}(S)$, for $H_i \equiv H_t$ and so from the fact, that in every point of the convex set S, there exists a supporting hyperplane to it). This means, that $T^{-1}(\overset{\frown}{h_a h_b}) = \mathbf{H}_{ab}$ will be a set of supporting hyperplanes to S, which gives in $\boldsymbol{E}^n$ a connected set of their supporting points: $X_{ab} = \{\mathbf{x}_t : \mathbf{x}_t \in H_t \cap \text{Fr}(S) \neq \emptyset, H_t \cap \text{Int}(S) = \emptyset; H_t \subset \mathbf{H}_{ab}\}$. But by assumption the arc $\overset{\frown}{h_a h_b} \subset \mathsf{L}^n$ does not contain points of the set $\mathsf{H}_z = \{h_z\}$: $\{h_z\} \cap \overset{\frown}{h_a h_b} = \emptyset$, i.e.

$$T^{-1}(\mathsf{H}_z) \cap T^{-1}(\overset{\frown}{h_a h_b}) = \emptyset \Rightarrow \mathsf{H}_z \cap \mathbf{H}_{ab} = \emptyset.$$

But as soon as $\mathsf{H}_z \cap \mathbf{H}_{ab} = \emptyset$, then no hyperplane H_z exists such that, the point $\mathbf{x}_z \in H_z$ will fulfill the condition $\mathbf{x}_z \in X_{ab}$, where $\mathbf{x}_z (\mathbf{x}_z \in \boldsymbol{Z})$ is a supporting point for the hyperplane H_z to the set S. Evidently, the connected set X_{ab} is (or in the more common case – holds) the arc $\overset{\frown}{\mathbf{x}_a \mathbf{x}_b}$, where $\mathbf{x}_a \subset P_s$, $\mathbf{x}_b \subset P$ and should fulfill the condition: $\overset{\frown}{\mathbf{x}_a \mathbf{x}_b} \cap \boldsymbol{Z} = \emptyset$. This condition will contradict to the Property 2.3, because $\boldsymbol{Z}$ is a common contour of the sets P_s and P, which means that the initial assumption: $\mathsf{H}_z \cap \overset{\frown}{h_a h_b} = \emptyset$, in the space L^n, is incorrect – i.e. the condition $\mathsf{H}_z \cap \overset{\frown}{h_a h_b} \neq \emptyset$ will be fulfilled. Then according to Definition 2.1, the set H_z will be a common contour of the sets H_s and H_p in the space L^n, which completely proves Theorem 3.1.■

Appendix B

P r o o f of **Statement 3.1.** Let in the space $\boldsymbol{E}^n$, the axis $\boldsymbol{Y}$ crosses the hyperplane P in the point $\mathbf{x}_0 = (\mathbf{0}, y_0) \notin H_z$, which defines the hyperplanes $\{H_0\}$, supporting to P in this point: $\{H_0\} = \mathbf{H}_0 = \{H_0 \subset \boldsymbol{E}^n : H_0 \cap \mathbf{x}_0; H_0 \cap \text{Fr}(S) \neq \emptyset, H_0 \cap \text{Int}(S) = \emptyset\}$. The set $\mathbf{H}_0$ may consist of only one hyperplane and evidently: $\mathbf{H}_0 \subset \mathsf{H}_p$. The transform $T(\mathbf{H}_0)$ in the space L^n will be the set $\boldsymbol{H}_0$ (which may consist of only one single point too). Since in $\boldsymbol{E}^n$ all the hyperplanes $\{H_0\}$ cross the axis $\boldsymbol{Y}$ in any of the same points $\mathbf{x}_0 = y_0$, then the image $T(\mathbf{H}_0) = \boldsymbol{H}_0$ in the space L^n, according to Lemma 1 [8], will be a set in the hyperplane $\mathbf{H}_0$, which is parallel to the subspace C^{n-1}, bearing in mind the equation: $\mathbf{x}_0 = (x_1 = 0, x_2 = 0, \ldots, x_{n-1} = 0) = \mathbf{0}$. Besides that, by condition we have: $\boldsymbol{x}_0 = P \cap \boldsymbol{Y} \neq \varnothing \Rightarrow \mathbf{x}_0 \in P$. This means that in $\mathbf{L}^n$ we will have $\boldsymbol{H}_0 \subset \mathbf{H}_{S2} \Rightarrow \boldsymbol{H}_0 = \mathbf{H}_0 \cap \mathbf{H}_{S2}$ and since $\mathbf{H}_0$ consists completely of supporting hyperplanes to P, then: $\boldsymbol{H}_0 \cap \mathrm{Int}(\mathbf{H}_{S2}) = \varnothing$, i.e. the set $\boldsymbol{H}_0$ will be a part of the hyperplane $\mathbf{H}_0$, supporting to $\mathbf{H}_{S2}$, which is parallel to the subspace C^{n-1}. From this it follows, that $\boldsymbol{H}_0$ consists completely of the points of the extremum $\{h_0\} = \{h_0: h_0 = [\boldsymbol{c}_0, f_0(\boldsymbol{c}_0)]: f_0(\boldsymbol{c}_0) = \min_c f(\boldsymbol{c})\}$, where $f(\boldsymbol{c})$ is the function specifying the hyperplane $\mathbf{H}_{S2}$, $f_0(\boldsymbol{c}_0) = L_0$, $(h_0 = (\boldsymbol{c}_0, L_0))$. Let us consider the set $\mathbf{H}_z = T(\mathsf{H}_z)$ too. Since the set $\mathsf{H}_z \subset \boldsymbol{E}^n$ is composed of the supporting hyperplanes H_z, setting the boundary of the cone with an apex at the point $\mathbf{x}_{z0} = (\mathbf{0}, y_{z0}) \in \boldsymbol{Y}(\mathbf{x}_{z0} \notin P)$, where for each hyperplane $H_z \subset \mathsf{H}_z$ we have $H_z \cap \boldsymbol{Y} = \mathbf{x}_{z0}$, then the set $\mathbf{H}_z$ in the space $\mathbf{L}^n$ will be the cross-section: $\mathbf{H}_z = \mathbf{H}'_z \cap \mathbf{H}_{S2}$, which is in the hyperplane $\mathbf{H}'_z$, parallel to the subspace C^{n-1} and situated in this subspace at a distance $L_z = y_{z0}$. It is clear that for the convex function $f(\boldsymbol{c})$, the set $\mathsf{C}_z = \{\boldsymbol{c}_z \in C^{n-1}: f(\boldsymbol{c}_z) \leq L_z\} = \mathrm{Pr}_C(\mathbf{I}_p)$ will be convex too, where $\mathbf{I}_p$ is an internal set related to the contour $\mathbf{H}_z \subset \mathbf{I}_p$. From the condition, which determines the set C_z, i.e. $L_0 < L_z$, for every extreme point $\boldsymbol{c}_0$ we will have: $\forall \boldsymbol{c}_0 \in \mathsf{C}_z \Rightarrow \{\boldsymbol{c}_0\} \subset \mathsf{C}_z$.

Let us consider a given point $\boldsymbol{c}_0^*$ in the set of the extreme points $\{\boldsymbol{c}_0\}$ and define the intercept of a straight line $\overline{\boldsymbol{c}_1\boldsymbol{c}_2} \ni \boldsymbol{c}_0^*$ with borders – the points $\boldsymbol{c}_1$, $\boldsymbol{c}_2 \in \mathrm{Fr}(\mathsf{C}_z)$. Obviously (in view of the convexity of the set C_z) will be obtained: $\overline{\boldsymbol{c}_1\boldsymbol{c}_2} \subset \mathsf{C}_z$. Then, bearing in mind, that the function $f(\boldsymbol{c})$ is convex, for the intercept of the straight line $\overline{\boldsymbol{c}_1\boldsymbol{c}_2}$ the inequality: $f\,[\lambda\boldsymbol{c}_1 + (1-\lambda)\boldsymbol{c}_2] \leq \lambda f(\boldsymbol{c}_1) + (1-\lambda)\,f(\boldsymbol{c}_2)$ may be written, where by condition: $\boldsymbol{c}_0^* = \lambda^*\boldsymbol{c}_1 + (1-\lambda^*)\boldsymbol{c}_2$, $\lambda^* \in \{\lambda : \lambda \in [0, 1]\}$. It is clear, that the intercept of the straight line $\overline{\boldsymbol{c}_1\boldsymbol{c}_2}$ will correspond to the intercept of the straight line $\overline{h_1 h_2} \subset \mathrm{epi}[f(\boldsymbol{c})]$, for $\lambda \in (0, 1)$ and $h_1 = [\boldsymbol{c}_1, f(\boldsymbol{c}_1)]$, $h_2 = [\boldsymbol{c}_2, f(\boldsymbol{c}_2)]$, which will define the arc $\overset{\frown}{h_1 h_2} \ni h_0^* = [\boldsymbol{c}_0^*, f(\boldsymbol{c}_0^*)]$, where $\overset{\frown}{h_1 h_2} = \{h_{12}^{\lambda} : h_{12}^{\lambda} = [\boldsymbol{c}_\lambda, f(\boldsymbol{c}_\lambda)], \lambda \in [0, 1]\}$, for $\boldsymbol{c}_\lambda = \lambda\boldsymbol{c}_1 + (1-\lambda)\boldsymbol{c}_2$. Since for the border points h_1 and h_2 we have h_1, $h_2 \in \mathbf{H}_z$ (because: $\boldsymbol{c}_1$, $\boldsymbol{c}_2 \in \mathrm{Fr}(\mathsf{C}_z)$), then $\overset{\frown}{h_1 h_2} \subset \mathbf{I}_p$ and obviously $h_0^* \in \mathbf{I}_p$ (more accurately: $h_0^* \in \mathrm{Int}(\mathbf{I}_p)$, because $h_0^* \notin \mathbf{H}_z$). Then the set of the arcs $\bigcup_i \overset{\frown}{h_{1i} h_{2i}}$, with border points h_{1i} and h_{2i} will form a covering of the internal set $\mathbf{I}_p$, towards the contour $\mathbf{H}_z$: $\bigcup_i \overset{\frown}{h_{1i} h_{2i}} = \mathbf{I}_p$.

Let us take an internal random point $h_i \in \mathrm{Int}(\mathbf{I}_p)$ and define the arc $\overset{\frown}{h_0^* h_i}$, for which we obviously have: $\overset{\frown}{h_0^* h_i} \subset \mathrm{Int}(\mathbf{I}_p)$, i.e. $\overset{\frown}{h_0^* h_i} \cap \mathbf{H}_z = \varnothing$. Then $T^{-1}(\overset{\frown}{h_0^* h_i}) = \mathbf{H}_{0i}^*$ will be a set of hyperplanes, whose supporting points will not cross the contour $\boldsymbol{Z}$ in the space $\boldsymbol{E}^n$, i.e. $\mathbf{H}_{0i}^* \subset \mathrm{Int}(\mathsf{H}_p)$ or $\mathbf{H}_{0i}^* \subset \mathrm{Int}(\mathsf{H}_s)$. Since for $H_0^* = T^{-1}(h_0^*)$ we have by condition: $H_0^* \subset \mathbf{H}_0$ and $\mathbf{H}_0 \subset \mathrm{Int}(\mathsf{H}_p)$, where $H_0^* \subset \mathbf{H}_0$, then for the whole set $\mathbf{H}_{0i}^*$ we will have $\mathbf{H}_{0i}^* \subset \mathrm{Int}(\mathsf{H}_p)$. Obviously for their images in $\mathbf{L}^n$ we have $\overset{\frown}{h_0^* h_i} \subset \mathrm{Int}(\mathbf{H}_p)$, i.e. $\overset{\frown}{h_0^* h_i} \subset \mathbf{H}_p \cap \mathbf{I}_p \neq \varnothing$. Let us assume that $\mathrm{H}_z \subset \mathrm{H}_p$ in the space $\boldsymbol{E}^n$, from where in $\mathbf{L}^n$ we have: $\mathbf{H}_z \subset \mathbf{H}_p$. Then, for the set $\mathbf{H}_p$ there will exist the arcs $\overset{\frown}{h_{1i} h_{2i}}$, each of which contains the point: $h_0^*: h_0^* \in \forall \overset{\frown}{h_{1i} h_{2i}}$ and it is such that $\overset{\frown}{h_{1i} h_{2i}} \subset \mathbf{H}_p$, which means that for $\mathbf{H}_p$ we can form the covering: $\bigcup_i \overset{\frown}{h_{1i} h_{2i}} = \mathbf{H}_p$. Since for the border points h_{1i} and h_{2i} we have (by condition): h_{1i}, $h_{2i} \in \mathbf{H}_z$, then obviously (as was specified above) the arcs $\bigcup_i \overset{\frown}{h_{1i} h_{2i}}$ will also form the covering of the internal set $\mathbf{I}_p$, toward the contour $\mathbf{H}_z$: $\bigcup_i \overset{\frown}{h_{1i} h_{2i}} = \mathbf{I}_p$, from where it immediately follows that $\mathbf{I}_p = \mathbf{H}_p$, i.e. $\mathbf{H}_p$ is an internal set toward the contour $\mathbf{H}_z$. It is clear, that if the contour $\mathbf{H}_z$ partitions the hyperplane $\mathbf{H}_{S2}$, only into two sets $\mathbf{H}_s$ and $\mathbf{H}_p$, in the space $\mathbf{L}^n$, then for $\mathbf{H}_s$ we have the equation $\mathbf{H}_s = \mathbf{H}_{S2} \setminus \mathbf{H}_p$ which means, that $\mathbf{H}_s$ will be an external set in relation to $\mathbf{H}_z$, where $\mathbf{H}_s = T(\mathrm{H}_s)$. In this way the Statement 3.1 is completely proven ■

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