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Algorithms for Solving the Problem for a Minimal Network Circulation with One Side Constraint

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Abstract: *Two algorithms for solving the problem above are proposed. The first one use scaling technology and solve the continuous problem. The second use the method of min ratio canceling and may solve the integer problem.*

Keywords: *Flow with side constraints, bicriteria network flow, one side constraint, min ratio cancelin, scaling.*

1. Introduction

Consider a graph $G = \{N, U\}$ with n nodes and m arcs. With each arc $x_i \in U$ are associated two numbers c_i and b_i which practical means may be different – a cost and travel time for example.

The problem for minimal circulation $x = \{x_{i,}, x_i \in U\}$ with one side constraint (Cosc) may be defined as follows:

MC: $$\min cx = \sum_{x_i \in U} c_i x_i$$

subject to

(1) $$\sum_{x_j \in A(i)} x_j - \sum_{x_j \in B(i)} x_j = 0, \quad i \in N,$$

(2) $$bx = \sum_{x_i \in U} b_i x_i = b^0;$$

(3) $$0 \le x_i \le u_i, \quad x_i \in U.$$

We denote: by $A(i)$ and $B(i)$ the sets of forward and backward arcs incident to the node i, by X – the set of circulations x which satisfy the constraints (1) and (3).

The optimal solution of the problem MC is denoted by $c^{\text{opt}} = cx^{\text{opt}}$ and by b^* and B are denoted the minimum and the maximum of bx on X.

The defined problem is a generalization of the ordinary problem for minimal circulation on network which generalization is caused by theoretical and practical necessities. The problem arise defining scalarization problems of the BiCriteria network Flow (BCF). It is proved in [1, 2] that each efficient solution of BCF may be represented like a sum of an nondominated flow and an nondominated circulation in the incremental graph.

The problem MC is a linear programming problem, but the researcher is provoked to exploit the unimodulare structure of the matrix of the constraints (1) and (3). Many adapted versions of linear programming algorithms are proposed. But usually these are not polynomial and do not ensure the integrality of the optimal solution when the input data is integral. The optimal solution of the problem being investigated may be received by a specialization of the simplex algorithm, which exploits the embedded network (unimodular) structure in the constraint matrix. In [7] it is proven, that the basic solution of this problem has a structure that corresponds to a spanning tree with one additional arc in the graph G. In [8] three different methods (primal, dual and Lagrangean) are described and their computational efficiency is compared. These methods handle the side constraint within the framework of a network code. But none of them is polynomial.

In the present paper are proposed primal algorithms for solving the defined problem, which logic is based on the idea to improve the value of the objective, searching a feasible improving direction, solving appropriately defined subproblems. Such type of algorithms are used for solving polynomialy the min-cost flow problem[3], the problem for optimal submodular flow [4], the problem for minimum cost generalized flow [5], linear programs [6], certain classes of integer programs [7].

2. The continuous problem

Let y be a circulation in G. We denote by $G(y) = \{N, U_y\}$ the incremental graph for y and by c, b, u – the corresponding parameters of the arcs. It is is constructed in the following way:

– for each arc $i \in U$ an arc $j \in U(y)$ exists if $y_i < u_i$ and $c_j := c_i$, $b_j := b_i$, $u_j := u_i - y_i$;

– for each arc $i \in U$ an arc $j \in U(y)$ exists if $y_i > 0$ and $c_j := -c_i$, $b_j := -b_i$, $u_j := y_i$.

For each set I of arcs we denote by $b(x)$ or by bx the sum $\sum_{i \in I} b_i(x_i)$ of b-costs on these arcs, multiplied by the flow x on them, and by b – the sum of the b-costs.

For each sum b of b-costs on some arcs we denote its positive part by b^+ and by b^- – its negative part. The respective sums of the costs on these arcs are denoted by c, c^+ and c^-.

A circulation x in the residual network $G(y)$ is feasible if satisfies the conditions:

(4) $$\sum_{x_i \in U_y} b_i x_i = 0\,,$$

(5) $$0 \le x_i \le u_i,\ \ x_i \in U_y.$$

Note that if x is a feasible circulation in the incremental graph $G(y)$, then the circulation $z = y + x$, is a feasible circulation in the graph G.

The problem for optimal circulation in G(y) is denoted by IMC:

IMC: $$\min cx = \sum_{x_i \in U_y} c_i x_i$$

subject to (1), (4) and (5).

Lemma 1. The circulation y in G is optimal if and only if there is not a feasible negative cost circulation in the network $G(y)$.

P r o o f: If there is in G another circulation z with a lower cost, then the flow $z - y$ is a feasible one in the residual network and vise versa.

Lemma 2. If x is a circulation, then it is accomplished $cx=\sum \overset{+}{c}(y_1^{\,i}) + \sum \overset{-}{c}(y_2^{\,i})$ and $bx = \sum \overset{+}{b}(y_1^{\,i}) + \sum \overset{-}{b}(y_2^{\,i})$, where $y_1^{\,i}$ and $y_2^{\,i}$ are flows on cycles with positive and negative b-costs respectively.

P r o o f: Each circulation may be decomposed in cycles.

Let us define the problem IP on the incremental graph $G(y)$:

IP: $\min cx/bx$

subject to $x \in X$.

The problem IP is named "min-ratio circulation". We shell prove that the solution of IP if exists, is an improving direction for the flow y.

We will propose two algorithms, where in each iteration of whom, the gap between the cost of the flow on optimal circulation and the cost of the flow on the current circulation decreases until it becomes small enough. The integrality of the flow is not guaranteed.

Algorithm 1

Step 1. Let x^1 be an initial solution of the problem. $x := x^1$.

Step 2. Define $G(x)$. Formulate the problem IP.

Step 3. Find an optimal solution z of IP.

a) If $bz > 0$ and there is not a negative part in that sum, end. The flow x is optimal.

b) If $bz < 0$ and there is not a positive part in that sum, end. The flow x is optimal.

c) $bz = \overset{+}{b}(z^1) + \overset{-}{b}(z^2)$. If $bz=0$, $z^2 := pz^2$, $0 \le p < 1$. Determine numbers λ_1 and λ_2 such that

$\lambda_1 \overset{+}{b}(z^1) + \lambda_2 \overset{-}{b}(z^2) = \overset{+}{b}(\lambda_1 z^1) + \overset{-}{b}(\lambda_2 z^2) = 0.$

Set $z := (\lambda_1 z^1, \lambda_2 z^2)$; $x := x + z$; go to Step 2.

Theorem 1. The **Algorithm 1** finds an optimal solution of the problem **MC.**

P r o o f: Let z is an optimal solution of the problem IP defined in the Step 2. The circulation $x^{opt} - x$ is an feasible solution of the problem IP and because $bx^{opt} \ge 0$, it is accomplished $bz \ge 0$. Let $bz = \overset{+}{b}(z^1) + \overset{-}{b}(z^2)$, i.e. $\overset{+}{b}(z^1) \ge -\overset{-}{b}(z^2)$.

For $\lambda_1 \overset{+}{b}(z^1) + \lambda_2 \overset{-}{b}(z^2) = 0$ it is accomplished $\lambda_1 \le \lambda_2$ or $\lambda_2/\lambda_1 \ge 1$. Then:

$$\frac{\lambda_1 c^+(z^1) + \lambda_2 c^-(z^2)}{\lambda_1(b^+(z^1) + b^-(z^2))} = \frac{c^+(z^1) + \lambda_2 / \lambda_1 c^-(z^2)}{b^+(z^1) + b^-(z^2)} \le \frac{c^+(z^1) + c^-(z^2)}{b^+(z^1) + b^-(z^2)} \le \frac{c^{\text{opt}} - cx}{bx^{\text{opt}}}.$$

The last inequality is true when $x+z$ is not an optimal solution for MC.

$$\lambda_1 (\overset{+}{b}(z^1) + \overset{-}{b}(z^2)) = \lambda_1 (\overset{+}{b}(z^1) - \lambda_1/\lambda_2 \overset{+}{b}(z^1)) = \lambda_1(1-\lambda_1/\lambda_2)\overset{+}{b}(z^1)$$

The last is an feasible solution because $\lambda_1(1-\lambda_1/\lambda_2) \le 1$. Then:

$$\lambda_1 c^+(z^1) + \lambda_2 c^-(z^2) \le \frac{(c^{\text{opt}} - cx)\lambda_1(1-\lambda_1/\lambda_2)b^+(z^1)}{bx^{\text{opt}}} \le \frac{(c^{\text{opt}} - cx)b^*}{B}.$$

The last inequality implies that the flow on the current circulation $z = (\lambda_1 z^1, \lambda_2 z^2)$ improve the objective in every step with a fraction b^*/B from the best possible improvement.

The problem IP can be solved applying fractional programming binary search method for the parametric problem IPP:

IPP: $\min cx - \mu bx$

subject to $x \in X$.

The problem IPP is a minimal cost flow problem for fixed μ and may be solved effectively.

Algorithm 2

Step 1. Let x^1 be an initial solution of the problem; $x := x^1$.

Step 2. Define $G(x)$. Formulate the problem IP.

Step 3. Find two cycles σ_1 and σ_2 with costs $(\overset{+}{c_1}, \overset{+}{b_1})$ and $(\overset{+}{c_2}, \overset{+}{b_2})$ respectively, such that:

$\frac{c_1^-}{b_1^+} \le \frac{c_i^-}{b_i^+}$, $i \in \Phi_1$ (it is denoted by Φ_1 the set of all cycles in $G(x)$ with positive b-cost). The cycle σ_1 is a minimum mean cycle on Φ_1.

$\frac{c_2^-}{-b_2^-} \le \frac{c_i^-}{-b_i^-}$, $i \in \Phi_2$ (it is denoted by Φ_2 the set of all cycles in $G(x)$ with negative b-cost). The cycle σ_1 is a maximum mean cycle on Φ_2.

Step 4. Solve on the set of arcs defined by cycles σ_1 and σ_2

$\min (\overset{+}{c_1}(y) + \overset{-}{c_2}(y))$,

s.t. $\overset{+}{c}(y) + \overset{-}{c_2}(y) < 0$, (1) and (3)

a) If there is not a solution, end. The flow x is optimal.

Step 5. $bz = \overset{+}{b_1}(y^1) + \overset{-}{b_2}(y^2)$ where y^1 and y^2 are flows on these cycles determined in Step 4.

Determine numbers μ_1 and μ_2 such that:

$\mu_1 \overset{+}{b_1}(y^1) + \mu_2 \overset{-}{b_2}(y^2) = \overset{+}{b_1}(\mu_1 y^1) + \overset{-}{b_2}(\mu_2 z^2) = 0$

Set $z := (\mu_1 y^1, \mu_2 y^2)$; $x := x+z$; go to Step 2.

Theorem 2. The **Algorithm 2** finds an optimal solution of the problem **MC.**

P r o o f: Let cycles in $G(x)$ σ_1 and σ_2 are found in the Step 3. Let y is an optimal circulation for the problem IP. The circulation $x^{\text{opt}}-x$ is an feasible solution of the problem IP. It is accomplished:

$$\frac{c(y)}{b(y)} \le \frac{c^{\text{opt}} - cx}{bx^{\text{opt}}}.$$

Because bx^{opt} satisfies (2), it follows $b(y)\geq 0$, i. e. $\overset{+}{b}(y) \geq -\overset{-}{b}(y)$.

The flow y can be decomposed in flows on cycles. Let the flows y_1^i, $i\in I_k$, are the flows on cycles with positive b-costs and the flows y_2^i, $i\in I_l$, are the flows on cycles with negative b-costs. Let y^1 and y^2 are the flows determined in Step 5 of the Algorithm 2 on the cycle σ_1 – the minimum mean cycle on Φ_1 and on the cycle σ_2 – the maximum mean cycle on Φ_2, such that $\overset{+}{b_1}(y^1)+ \overset{-}{b_2}(y^2) \geq 0$ and $z^1=\mu_1 y^1$, $z^2= \mu_2 y^2$. (The case $\overset{+}{b_1}(y^1)+ \overset{-}{b_2}(y^2)\leq 0$ is analogous). The parameters μ_1 and μ_2 are defined from equality $\mu_1 \overset{+}{b_1}(y^1)+ \mu_2 \overset{-}{b_2}(y^2) = 0$, $\mu_2 \geq \mu_1$. It is clear that $\overset{-}{c_2}(y)\leq 0$, in the opposite case $\overset{+}{c_1}(y)$ would be less than $\overset{+}{c_1}(y)+ \overset{-}{c_2}(y)$. We define he numbers λ_1 and λ_2 such that

$$\lambda_1 \overset{+}{b}(y) + \lambda_2 \overset{-}{b}(y)=0,\ \lambda_2 \geq \lambda_1.$$

We choose a parameter p, $0 < p \leq 1$, such that $\lambda_1 \overset{+}{b_1}(z^1)+\lambda_2 \overset{-}{b_2}(pz^2)=0$, i.e. $p=\lambda_1/\lambda_2$. Than it is accomplished :

$$\frac{\sum_{i\in I_k} c^+(y_1^i) + \sum_{i\in I_l} c^-(y_2^i)}{\sum_{i\in I_k} b^+(y_1^i) + \sum_{i\in I_l} b^-(y_2^i)} = \frac{\sum_{i\in I_k} c^+(y_1^i) + \sum_{i\in I_l} c^-(y_2^i)}{(1-\lambda_1/\lambda_2)\sum_{i\in I_k} b^+(y_1^i)} = \frac{\sum_{i\in I_k} c^+(y_1^i)}{(1-\lambda_1/\lambda_2)\sum_{i\in I_k} b^+(y_1^i)} +$$

$$+\frac{\sum_{i\in I_l} c^-(y_2^i)}{(1-\lambda_1/\lambda_2)\sum_{i\in I_k} b^+(y_1^i)} = \lambda_1/(\lambda_2-\lambda_1)\left(\frac{\sum_{i\in I_k} c^+(y_1^i)}{\sum_{i\in I_k} b^+(y_1^i)} + \lambda_1/\lambda_2 \frac{\sum_{i\in I_l} c^-(y_2^i)}{\sum_{i\in I_l} -b^-(y_2^i)}\right) \geq$$

$$\text{(7)} \quad \geq \lambda_1/(\lambda_2-\lambda_1)\left(\frac{c_1^+(y_1)}{b_1^+(y_1)} + \lambda_1/\lambda_2 \frac{c_2^-(y_2)}{-b_2^-(y_2)}\right) =$$

$$\text{(8)} \quad = \lambda_1/(\lambda_2-\lambda_1)\left(\frac{c_1^+(z^1)}{b_1^+(z^1)} + \lambda_1/\lambda_2 \frac{c_2^-(pz^2)}{-b_2^-(pz^2)}\right) =$$

$$= \lambda_1/(\lambda_2-\lambda_1)\left(\frac{c_1^+(z^1)}{b_1^+(z^1)} + \lambda_1/\lambda_2 \frac{c_2^-(pz^2)}{\lambda_1/\lambda_2 b_1^+(z^1)}\right) =$$

$$= \lambda_1/(\lambda_2-\lambda_1)\left(\frac{c_1^+(z^1)+c_2^-(pz^2)}{b_1^+(z^1)}\right).$$

The inequality (7) is due to the fact that [10]:

$$\frac{a_1+\ldots+a_k}{g_1+\ldots+g_k} \geq \min_{i\in I_k} \frac{a_i}{g_i} \qquad \frac{a_1+\ldots+a_k}{g_1+\ldots+g_k} \geq \min_{i\in I_k} \frac{a_i}{g_i}$$

for all a_i and $g_i > 0,\ i \in I_k$; for some a_i <0 and $g_i > 0,\ i \in I_k$.

And from (6) it is follows:

$$\min(c_1^+(z^1)+c_2^-(z^2)) \leq c_1^+(z^1)+c_2^-(pz^2) \leq$$

(9) $$\leq \frac{(c^{\text{opt}}-cx)(\lambda_2/\lambda_1-1)b_1^+(z^1)}{bx^{\text{opt}}} \leq (c^{\text{opt}}-cx)\frac{b^{*2}}{B^2}$$

because $-\overset{-}{b}(y) \leq \overset{+}{b}(y) \leq B$ and

$(\lambda_2/\lambda_1 - 1)b_1^{+}(z^1)=((\overset{+}{b}(y)/(-\overset{-}{b}(y)) - 1)b_1^{+}(z^1)= b(y)b_1^{+}(z^1)/(-\overset{-}{b}(y)) \geq b^{*2}/B.$

This implies that the flow on the current two cycles $z = (\lambda_1 z^1, \lambda_2 z^2)$ improves the objective in every step with a fraction b^{*2}/B^2 from the best possible improvement.

3. The integer problem

Investigating this problem we solve the problem

MI: $$\min cx = \sum_{x_i\in U} c_i x_i$$

subject to (1), (3) and

(10) $$bx = \sum_{x_i\in U} b_i x_i \leq b^0;$$

$$x - \text{integer},$$
$$\Delta bx = b^0 - bx.$$

We propose the **Algorithm 3** for solving MI.

Algorithm 3

Step 1. Let x^1 be an initial solution of the problem; $x := x^1$.

Step 2. Define $G(x)$. Formulate the problem IP.

Step 3. Find two cycles σ_1 and σ_2 with costs (c_1^{+}, b_1^{+}) and (c_2^{+}, b_2^{+}) respectively, such that:

$\frac{c_1^-}{b_1^+} \leq \frac{c_i^-}{b_i^+}$, $i\in \Phi_1$ (it is denoted by Φ_1 the set of all cycles in $G(x)$ with positive b-cost). The cycle σ_1 is a minimum mean cycle on Φ_1.

$\frac{c_2^-}{-b_2^-} \leq \frac{c_i^-}{-b_i^-}$, $i\in \Phi_2$ (it is denoted by Φ_2 the set of all cycles in $G(x)$ with negative b-cost). The cycle σ_1 is a maximum mean cycle on Φ_2.

Step 4. Solve the defined bellow problem, on the set of arcs determined by cycles σ_1 and σ_2

min $(c_1^{+}(y)+ c_2^{-}(y))$,

s.t. $c_1^{+}(y)+ c_2^{-}(y) < 0,\ -bx \leq b_1^{+}(y)+ b_2^{-}(y) \leq \Delta bx,$ (1), (4) and (5)

a) If there is not a solution, end. The flow x is optimal.

Step 5. $bz = \overset{+}{b_1}(y^1) + \overset{-}{b_2}(y^2)$ where $y^1$and y^2 are flows on these cycles determined in Step 4.

Set $z := (y^1, y^2)$; $x: = x+z$; $bx: = bx+bz$; go to Step 2.

Theorem 3. The **Algoritm 3** finds an optimal solution of the problem **MI.**

P r o o f: It is similar to the proof or Theorem 2.

We note a fact that in (9) for $p \le 1$:

a) if $\overset{+}{c_1}(z^1)<0$ and $\overset{-}{c_2}(z^2)<0$, then $(\overset{+}{c_1}(z^1)+\overset{-}{c_2}(z^2)) \le \overset{+}{c_1}(z^1)+\overset{-}{c_2}(pz^2)$;

b) if: $\overset{+}{c_1}(z^1)<0$ and $\overset{-}{c_2}(z^2)>0$, then $\overset{+}{c_1}(z^1) \le \overset{+}{c_1}(z^1)+\overset{-}{c_2}(pz^2)$;

or $\overset{+}{c_1}(z^1)>0$ and $\overset{-}{c_2}(z^2)<0$, then $\overset{+}{c_1}(z^2) \le \overset{+}{c_1}(pz^1)+\overset{-}{c_2}(z^2)$.

There is a flow only on one of the cycles σ_1 and σ_2.

4. Conclusion

The cycles σ_1 and σ_2 may be determined using algorithms for min cost to time ratio cycle in the graphs with costs of arcs c and b-costs b and $-b$ respectively.

The number of iterations of the **Algorithm 1** is $O(B/b^*\log d)$ and of **Algorithms 2** and **3** is $O(B^2/b^{*2}\log d)$, where d is a bound of the length of cx^{opt}.

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